Classification of Cohen–Macaulay Modules on Plane Rational Cubics

Seminar talk – Definitions and Theorems

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1. Introduction

All elements $\mathcal{F}$ of the Simpson moduli space $M_{3m+1}(\mathbb{P}_3)$ of semi-stable coherent sheaves with Hilbert-polynomial $3m+1$ are Cohen-Macaulay modules. Among the $\mathcal{F} \in M_{3m+1}(\mathbb{P}_3)$, one finds sheaves which are supported on the plane irreducible rational cubics: the cuspidal and the nodal cubic.
We show that the "true" Cohen-Macaulay modules on the cuspidal cubic can be classified by their degree. Furthermore, they are reflexive sheaves.

2. Preliminaries

Let $k = \overline{k}$ and $X \subset \mathbb{P}_m(k)$ be a smooth projective variety of $\dim(X) = n$ and $\mathcal{O}_X(1)$ be an ample line bundle on X. Let $\mathcal{F} \in \mathrm{Coh}(X)$ denote a coherent sheaf on X.
Note: We can consider $\mathrm{Supp}(\mathcal{F})$ as a subscheme of X via

$$(Y, \mathcal{O}_Y) := (V(Ann(\mathcal{F})), i^*(\mathcal{O}_X/Ann(\mathcal{F})) \longrightarrow (X, \mathcal{O}_X)$$

where $\mathrm{Supp}(\mathcal{F}) = V(Ann(\mathcal{F})) \xrightarrow{i} X$ is a closed embedding.

Definition 2.1.
- $P_{\mathcal{F}}(m) := \chi(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)) := \chi(\mathcal{F}(m)) := \sum_{\nu=0}^{n} (-1)^\nu h^\nu(X, \mathcal{F}(m)) = \frac{a_d}{d!} m^d + \frac{a_{d-1}}{(d-1)!} m^{d-1} + \cdots + a_0$ is the *Hilbert-polynomial* of $\mathcal{F}$. $\dim(\mathcal{F}) := \dim(\mathrm{Supp}(\mathcal{F})) = d$ is the *dimension* of $\mathcal{F}$. $\mu(\mathcal{F}) := a_d$ is called the *multiplicity* of $\mathcal{F}$.
- We say $\mathcal{F}$ is *pure-dimensional* of $\dim(\mathcal{F}) = d$ iff for all coherent subsheaves $0 \neq \mathcal{F}' \subset \mathcal{F}$ $\dim(\mathcal{F}') = d$.
- A pure-dimensional sheaf $\mathcal{F}$ of dimension d is called
  - *semi-stable* iff for all coherent submodules $0 \neq \mathcal{F}' \subset \mathcal{F}$
  $$\frac{P_{\mathcal{F}'}}{\mu(\mathcal{F}')} \leq \frac{P_{\mathcal{F}}}{\mu(\mathcal{F})}$$
  (Note: $\leq$ is here the lexicographical ordering on $\mathbb{Q}[m]$)
  - *stable* iff for all coherent submodules $0 \neq \mathcal{F}' \subset \mathcal{F}$
  $$\frac{P_{\mathcal{F}'}}{\mu(\mathcal{F}')} < \frac{P_{\mathcal{F}}}{\mu(\mathcal{F})}$$
- $\mathcal{F} \in \mathrm{Coh}(X)$ is a *Cohen-Macaulay module* iff $\mathrm{prof}_{\mathcal{O}_{X,p}} \mathcal{F}_p = \dim(\mathcal{F}_p)$ for all $p \in \mathrm{Supp}(\mathcal{F})$.

Remarks

- $\mathcal{F}$ is pure-dimensional of $\dim(\mathcal{F}) = d$ iff all associated points $p \in \mathrm{Ass}(\mathcal{F})$ have dimension d.
- If $\mathcal{F} \cong \mathcal{O}_Z$ where $Z \subset X$ is a subscheme of $\dim Z = d$ we have:
  $\mathcal{O}_Z$ is pure-dimensional $\Longleftrightarrow$ Z has no zerodimensional (possibly embedded) components and all components of of dimension $\in (0, d)$ are embedded.

Examples:

Lemma 2.1. Let $\mathcal{F} \in \mathrm{Coh}(X)$ be pure dimensional of dimension $\dim(\mathcal{F}) = d$. For all $p \in \mathrm{Supp}(\mathcal{F})$ with $\mathrm{ht}(p) > n - d$ it follows that $\mathrm{prof}_{\mathcal{O}_{X,p}} \mathcal{F}_p \geq 1$

Corollary 2.1. If $\mathcal{F}$ is a pure coherent sheaf of $\dim(\mathcal{F}) = d$ on the projective variety X, $\dim(X) = n$ then

(i) The stalks $\mathcal{F}_p$ are Cohen-Macaulay for all $p \in \mathrm{Supp}(\mathcal{F})$ with $\mathrm{ht}(p) = n - d + 1$.
(ii) $Z := \{x \in X \,|\, \mathcal{F}_x$ is not Cohen-Macaulay $\}$ is a closed subvariety of X and $\dim(Z) \leq d - 2$.

In the proof of Corollary 2.1,(ii) we'll use the following "tools":

Let $\mathcal{F} \in \mathrm{Coh}(X)$. We define the sets:

$$S_m(\mathcal{F}) := \{x \in X \,|\, \mathrm{prof}_{\mathcal{O}_X,x} \mathcal{F}_x \leq m\} = \bigcup_{r \geq n-m} \mathrm{Supp}(\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X))$$

and

$$D_m(\mathcal{F}) := \{x \in \mathrm{Supp}(\mathcal{F}) \,|\, \dim(\mathcal{F}_x) - \mathrm{prof}_{\mathcal{O}_X,x} \mathcal{F}_x \geq m\}, \quad m \geq 1$$

Note that $D_1(\mathcal{F}) = \{x \in X \,|\, \dim(\mathcal{F}_x) \neq \mathrm{prof}_{\mathcal{O}_X,x} \mathcal{F}_x\}$ is the set where the stalks of $\mathcal{F}$ are not Cohen-Macaulay.

Lemma 2.2. Let $\mathcal{F} \in \mathrm{Coh}(X)$. The sets $S_m(\mathcal{F})$ and $D_m(\mathcal{F})$ are closed subvarieties of X. Moreover, $\dim(S_m(\mathcal{F})) \leq m$ and $\mathrm{codim}_{\mathrm{Supp}(\mathcal{F})}(D_m(\mathcal{F})) \geq m$.

Theorem 2.1. (Scheja/Trautmann) Let X be an n-dimensional projective variety (or complex space), $Z \subset X$ a closed subscheme defined by the ideal sheaf $\mathcal{I}$ and $\mathcal{F} \in \mathrm{Coh}(X)$. Then for all $q \geq 0$ the following conditions are equivalent:

(i) $\inf_{x \in Z} \mathrm{prof}_{\mathcal{I}_x} \mathcal{F}_x \geq q + 1$.
(ii) $\dim(Z \cap S_{k+q+1}(\mathcal{F})) \leq k$ for any k.
(iii) $\mathcal{H}^i_Z(\mathcal{F}) = 0$ for $i \leq q$.
(iv) For any open subset U of X the restrictions

$$H^i(U, \mathcal{F}) \longrightarrow H^i(U \setminus Z, \mathcal{F})$$

are bijective for $i < q$ and injective for $i = q$.

Proof: cf. Bănică, [1], p.67

Examples:

We can formulate the following

Theorem 2.2. Let $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}_n)$ be semi-stable with Hilbert-polynomial $P_{\mathcal{F}}(m) = am + b$. Then $\mathcal{F}$ is a Cohen-Macaulay module.

From now on fix a plane $\mathbb{P}_2 \cong H \subset \mathbb{P}_3$. We can characterize stable sheaves $\mathcal{F}$ supported in $\mathbb{P}_2$ with $P_{\mathcal{F}}(m) = 3m + 1$ by extensions:

Lemma 2.3. Let $\mathcal{F}$ be a sheaf as described above. Then there is a plane cubic $C \subset \mathbb{P}_2$ and a point $p \in C$ such that the $\mathcal{O}_C$–module $\mathcal{F}$ has an extension of the form:

$$0 \to \mathcal{O}_C \to \mathcal{F} \to k_p \to 0 \qquad (*)$$

where k_p is the skyscraper sheaf at the point p.

Remark: Because of Theorem 2.2, the sheaves $\mathcal{F}$ are Cohen-Macauley. Thus, the extensions $(*)$ are non-split: By definition $1 = \mathrm{depth}(\mathcal{F}_x) = \inf\{i : \mathrm{Ext}^i_{\mathcal{O}_{C,x}}(k, \mathcal{F}_x) \neq 0\}$. Therefore $\mathrm{Hom}_{\mathcal{O}_{C,x}}(k, \mathcal{F}_x) = 0 \; \forall x \in \mathrm{Supp}(\mathcal{F})$. It follows that $\mathcal{H}om_{\mathcal{O}_C}(k_p, \mathcal{G}) \cong 0$ and in particular $\mathrm{Hom}(C, k_p, \mathcal{F}) = 0$. So we get an injection $0 \neq \mathrm{Hom}(k_p, k_p) \to \mathrm{Ext}^1(k_p, \mathcal{O}_C)$. This implies that $\mathrm{Ext}^1(k_p, \mathcal{O}_C) \neq 0$.
In particular, we get $p \in C$.

Lemma 2.4. Let $C \subset \mathbb{P}_2$ be a plane cubic curve, $\mathcal{F}$ be a a pure-dimensional $\mathcal{O}_C$–module. Then are equivalent:

(i) There exists an extension of type $(*)$.
(ii) $\mathcal{F}$ has the following free resolution in $\mathbb{P}_2$:

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-2) \underset{\left(\begin{smallmatrix} q_1 & w_1 \\ q_2 & w_2 \end{smallmatrix}\right)}{\longrightarrow} \mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \to \mathcal{F} \to 0, \qquad (**)$$

where $w_1, w_2 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ are independent linear generators of the ideal sheaf m_p and $q_1, q_2 \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2))$ are quadratic forms such that $q_1 w_2 - q_2 w_1$ is the equation of the cubic C. (The matrix $\left(\begin{smallmatrix} q_1 & w_1 \\ q_2 & w_2 \end{smallmatrix}\right)$ is unique up to the choice of w_1, w_2.)

Remark: If w_1 and w_2 are linear dependent then $\mathcal{F}$ is not semi-stable because in this case $\left(\begin{smallmatrix} q_1 & w_1 \\ q_2 & w_2 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} \tilde{q}_1 & 0 \\ q_2 & w_2 \end{smallmatrix}\right)$. The corresponding extension becomes $0 \to \mathcal{O}_{\tilde{C}} \to \mathcal{F} \to \mathcal{O}_L \to 0$, where $\tilde{C}$ is the conic defined by the form $\tilde{q}_1$ and L is the line defined by w_1.

3. What is a moduli problem?

Moduli problems usually arise in connection with classification problems. One has a certain set of objects A equipped with an equivalence relation $\sim$. A is "related to a certain category" – let's say to complex manifolds with holomorphic maps as morphisms. (cf. example 1).
The aim is to give the set $A/\sim$ the structure of an object in this category – in this example the structure of a comlpex manifold.

Examples:

1. How many different holomorphic structures exist on a torus $\mathbb{C}/\Lambda$? Answer: As many as complex numbers. They form a complex manifold.
2. How many ellipitic curves (smooth $g = 1$ curves) exist? Answer: $M_g = \mathbb{A}^1_k$. What about higher genus curves? $\Rightarrow$ They form a "coarse" moduli space M_g which is an irreducible quasi-projective variety of dimension $3g - 3$ (Deligne/Mumford).
3. Let V be an n-dimensional k - vector space and
$$M_c := \{T \in \mathrm{End}(V) \,|\, \mathrm{T \text{ is cyclic, i.e. }} \{v, Tv, \ldots, T^{n-1}v\} \text{ is a basis of } V\}.$$
Then $M_c = \mathbb{A}^n_k$ is a "fine" moduli space.
4. A certain physicist had the following weird idea: Let (M^4, g) be a compact, connected, oriented, differentiable Riemannian 4-fold, $\sigma \in \mathrm{Spin}^c(M)$ be a Spin^c structure, $\mathcal{A}_\sigma$ the affine space of Hermitian connections on the associated complex line bundle $\det(\sigma)$. Moreover, let $\mathbb{S}^+_\sigma$ be the comlpex odd spinor bundle and $\mathcal{D}_A : \Gamma(\mathbb{S}^+_\sigma) \longrightarrow \Gamma(\mathbb{S}^-_\sigma)$ the geometric Dirac operator constructed via σ, the Clifford-multiplication and the connection on $\mathbb{S}^+_\sigma$ (the latter is induced by $A \in \mathcal{A}_\sigma$).
Furthermore, let $\mathcal{Z}^\eta_\sigma$ denote the set of solutions $(\phi, A) \in \Gamma(M, \mathbb{S}^+_\sigma) \times \mathcal{A}_\sigma$ of the following system of "non-linear PDEs":
$$\mathcal{D}_A(\phi) = 0 \text{ and } c(F^+_A + i\eta^+) = q(\phi)$$
Here, $c : i\Lambda^2_+ T^*M \longrightarrow \mathrm{End}(\mathbb{S}^+_\sigma)$ denotes the Clifford-multiplication, $F^+_A \in i\Omega^2_+(M)$ the self-dual part of the curvature of A and $\Gamma(\mathbb{S}^+_\sigma) \xrightarrow{q} \mathrm{End}(\mathbb{S}^+_\sigma)$, where $q(\phi) := \overline{\phi} \otimes \phi - \frac{1}{2}|\phi|^2 \mathrm{id}$. We define $M(g, \sigma, \eta) := \mathcal{Z}^\eta_\sigma / \mathrm{Aut}(\det(\sigma))$. Is this set a moduli space? Witten proved that $M(g, \sigma, \eta)$ is a compact, oriented, differentiable manifold for "good" $\eta^+ \in \Gamma(\Lambda^2_+ T^*M)$.
5. Fix an n-dimensional k - vector space V. Let's consider families of quotient vector bundles $\mathcal{E}$ of rank $r \leq n$ over a base scheme S:
$$Grass^r_V(S) := \{V \otimes \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow 0\}$$
Then the contravariant functor $Grass^r_V(\cdot)$ is represented by a "fine" moduli space Grass^r_V, the "usual" Grassmannian which carries obviously the structure of a projective variety.

6. More generally, one can consider the Hilbert- (or "Quot-") scheme $\mathrm{Hilb}^P_X(\mathcal{E})$ of isomorphism classes of coherent quotients $\mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$ of a locally free sheaf $\mathcal{E}$ on a smooth projective variety X with fixed Hilbert-polynomial $P_{\mathcal{G}} = P$.
   Grothendieck showed that it is a "fine" moduli space and carries the structure of a projective variety.
   For $\mathcal{E} = \mathcal{O}_{\mathbb{P}_n}$ this is the "classical" Hilbert-scheme of subschemes $Y \subset \mathbb{P}_n$ with fixed Hilbert-polynomial.

Now we'd like to give precise definitions of what we mean by a "fine" or "coarse" moduli space in the category of algebraic varieties:

Definition 3.1. We consider a collection of objects A together with an equivalence relation $\sim$.

- A *family* X of objects of A parametrized by a base-scheme S shall fullfill the following conditions:
  1. If X is parametrized by a point $\{pt\}$ then X is a single object of A.
  2. For every base scheme S, there is a notion of equivalence of families X parametized by S which reduces to $\sim$ for $S = \{pt\}$. We denote these relations also by $\sim$.
  3. For any morphism $\phi : S' \longrightarrow S$ and any family X parametrized by S there is an induced family $\phi^* X$ parametrized by S'. This operation satisfies the following properties: $X \sim X' \Rightarrow \phi^* X \sim \phi^* X'$, $id_S^* =$ identity and $(\phi \circ \phi')^* = \phi'^* \circ \phi^*$.

  Let $\mathcal{Z}(S)$ denote the set of equivalence classes of families X parametrized by a base-scheme S.
  Condition 3 says that $\mathcal{Z}(\cdot) : (schemes) \longrightarrow (sets)$ is a contravariant functor.
- A *moduli problem* consists of the set A, the equivalence relation $\sim$ and a functor $\mathcal{Z}(\cdot)$ "of families" in A.
- A *fine moduli space* for a given moduli problem is a pair (M, Φ) which represents the functor $\mathcal{Z}$. (i.e. there is a variety M and a natural transformation of contravariant functors $\Phi : \mathcal{Z}(\cdot) \to \mathrm{Mor}(\cdot, M)$ that is an equivalence of functors.)
- A *coarse moduli space* for a given moduli problem is a variety M together with a natural transformation of functors $\Phi : \mathcal{Z}(\cdot) \to \mathrm{Mor}(\cdot, M)$ such that:
  - $\Phi(pt)$ is a bijection.
  - (Universal property) For any variety N and any natural transformation $\Psi : \mathcal{Z}(\cdot) \to \mathrm{Mor}(\cdot, M)$, there exists a unique natural transformation $\Omega : \mathrm{Mor}(\cdot, M) \to \mathrm{Mor}(\cdot, N)$ such that $\Psi = \Omega \circ \Phi$.

If one wants to describe the moduli space of bundles or coherent sheaves on a smooth projective variety, it turns out that it is not enough to consider flat families with fixed Hilbert-polynomial (they may not even be bounded: consider the family of rank r and degree d bundles $\{\mathcal{O}_X(-ka) \oplus \mathcal{O}_X((d+k)a) \oplus (r-2)\mathcal{O}_X\}_{k \in \mathbb{Z}}$ on a smooth curve X, $a \in X$). Therefore, we have to impose more conditions on the families. One possibility to do this is to consider only families of semi-stable sheaves:

Let X be a smooth projective variety and P a fixed polynomial. Define the contravariant functor $\underline{M}_P(X)(\cdot)$: $(schemes) \longrightarrow (sets)$, where

$$\underline{M}_P(X)(S) := \{\mathcal{F} \in \mathrm{Coh}(S \times X) \,|\, \mathcal{F} \text{ is } S\text{-flat}, \mathcal{F}(s) \text{ is semi-stable } \forall\, s \in S \text{ closed, and } P_{\mathcal{F}(s)} = P\}$$

and $\forall\, \psi : S' \longrightarrow S$:

$$\underline{M}_P(X)(\psi) : \underline{M}_P(X)(S) \longrightarrow \underline{M}_P(X)(S')\,; \quad \mathcal{F} \mapsto (\psi \times \mathrm{id})^*(\mathcal{F})$$

Theorem 3.1. (Simpson, 1994) There is a coarse moduli space $M_P(X)$ for the functor $\underline{M}_P(X)(\cdot)$ with the following properties:

- The algebraic variety $M_P(X)$ is projective.
- The closed points of $M_P(X)$ are the "S-equivalence classes" of semi-stable coherent sheaves $\mathcal{F}$ on X with Hilbert-polynomial P.
- The set of isomorphism classes of semi-stable sheaves with fixed P is an open subset of $M_P(X)$.

Proof: In my next talk...

Conjecture (Trautmann, Freiermuth) Let H_0 be the component of $\mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$ containing the twisted cubics. The Simpson moduli space $M_{3m+1}(\mathbb{P}_3)$ consists of at least two irreducible, smooth components M_0 and M_1. $M_0 \cong H_0$ and is of dimension 12. M_1 consists of sheaves $\mathcal{F}$ with planar support which have an extension of type $(*)$. The dimension of M_1 is $13 (= 3 + 9 + 1)$. The intersection $M_1 \cap M_2$ is transversal and its elements are true Cohen-Macaulay modules $\mathcal{F}$ with singular support C which are not free at a singular point of C.

4. Cohen-Macauley modules on the plane cuspidal cubic

4.1. **Basic Properties of the cuspidal cubic C.** Let $C \subset \mathbb{P}_2(k)$ be a cuspidal cubic given by the ideal $I_C = (z_0 z_1^2 - z_2^3)$. $\mathcal{I}_C := \tilde{I_C}$. Note that $H^0_*(\mathcal{I}_C) = I_C$, i.e. I_C is saturated. C has a singularity at $p = \langle 1, 0, 0 \rangle$. The parametrization ν:

$$L := \mathbb{P}_1 \xrightarrow{\nu} C \quad \langle s,t\rangle \mapsto \langle s^3, t^3, st^2\rangle \quad \text{with} \quad \mathcal{O}_C \xrightarrow{\nu^*} \nu_*\mathcal{O}_L$$

is a bijective, finite morphism. The cokernel $\mathcal{C}$ in the exact sequence $0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_L \to \mathcal{C} \to 0$ has length 1 and is therefore isomorphic to k_p. This can be checked by looking at the Hilbert-polynomials $P_{\mathcal{O}_C}$ and $P_{\nu_*\mathcal{O}_L}$: From $0 \to \mathcal{O}(-3) \to \mathcal{O} \to \mathcal{O}_C \to 0$ on $\mathbb{P}_2$ (α) we get $P_{\mathcal{O}_C}(m) = 3m$.
$\nu^*(\mathcal{O}_{\mathbb{P}_2}(d)|_C) \cong \mathcal{O}_L(3d)$ and the projection formula give

$$P_{\nu_*\mathcal{O}_L}(m) = \chi((\nu_*\mathcal{O}_L)(m)) = \chi(\nu_*(\mathcal{O}_L \otimes_{\mathcal{O}_L} \nu^*(\mathcal{O}_{\mathbb{P}_2}(m)|_C))) = \chi(\nu_*(\mathcal{O}_L(3m)) = \chi(\mathcal{O}_L(3m)) = 3m+1$$

Therefore, we get an exact sequence

$$0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_L \to k_p \to 0$$

Let's mention some further properties of C:

- From the sequence (α) we see that C is arithmetically Gorenstein and thus $\mathcal{O}_C \cong \omega_C(2+1-3) = \omega_C$. We also have $\operatorname{reg}(\mathcal{O}_C) = 2$.
- $h^1(C, \mathcal{O}_C) = 1$, $h^0(C, \mathcal{O}_C) = 1$, $\chi(\mathcal{O}_C) = 0$
- Since C is integral, $p_a(C) = h^1(C, \mathcal{O}_C) = 1$.

The above morphism ν ist the normalization and desingularization of C and induces an isomorphism

$$k \cong \mathbb{P}_1 \smallsetminus \{q\} \underset{\approx}{\xrightarrow{\nu}} C \smallsetminus \{p\} =: C^*.$$

where $q = \langle 1, 0\rangle \in L$. By means of that isomorphism C^* becomes an abelian group with group law

$$\langle s_1^3, 1, s_1\rangle + \langle s_2^3, 1, s_2\rangle = \langle (s_1+s_2)^3, 1, s_1+s_2\rangle.$$

We also write $x(s)$ for $\langle s^3, 1, s\rangle$.

Lemma 4.1. (i) Let x_1, x_2, x_3, pairwise different points on C^*. Then $x_1 + x_2 + x_3 = 0$ if and only if they are on a line.
(ii) $2x_1 + x_2 = 0$ if and only if x_2 is the intersection of C with the tangent line of C at x_1.
(iii) The zero element $0 = x(0)$ of C^* is the point $\langle 0, 1, 0\rangle$.

The degree of a line bundle $\mathcal{L}$ on C is defined as the degree of $\nu^*\mathcal{L}$, which is a bundle $\mathcal{O}_L(d)$. So

$$\deg \mathcal{L} = d \quad \text{iff} \quad \nu^*\mathcal{L} \cong \mathcal{O}_L(d).$$

The Riemann–Roch theorem in this case becomes

$$\chi\mathcal{L} = h^0\mathcal{L} - h^1\mathcal{L} = \deg \mathcal{L}$$

because $\chi\mathcal{O}_C = 0$.

4.2. **The Cohen-Macaulay modules $\nu_*\mathcal{O}_L(d)$.** For the normalization $L \xrightarrow{\nu} C$ of a plane cuspidal cubic $\nu_*\mathcal{O}_L(d)$ denotes $\nu_*(\mathcal{O}_L(d))$. From the extension

$$0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_L \to k_p \to 0 \qquad (i)$$

we see that the $\mathcal{O}_C$–module $\nu_*\mathcal{O}_L$ has rank 1 on C, but it is not free at p. Since C is integral, we immediately get from (i) that $\nu_*\mathcal{O}_L$ is stable. Since $P_{\nu_*\mathcal{O}_L}(m) = 3m+1$ we get from Theorem 2.2 that $\nu_*\mathcal{O}_L$ is Cohen-Macauley. It lies in the component M_1 of the moduli-space $M_{3m+1}(\mathbb{P}_3)$. From Lemma 2.4 we conclude that $\nu_*\mathcal{O}_L$ has a free resolution of the form:

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-2) \underset{A}{\longrightarrow} \mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \to \nu_*\mathcal{O}_L \to 0, \qquad (ii)$$

where $A = \begin{pmatrix} x_2^2 & z_1 \\ z_0 z_1 & z_2 \end{pmatrix}$. (We choose the forms z_1, z_2 as generators of m_p.)

Remarks

- We can also see from (ii) that $\nu_*\mathcal{O}_L$ is Cohen-Macauley. Auslander/Buchsbaum implies for all $x \in \operatorname{Supp}(\nu_*\mathcal{O}_L)$:
$$\operatorname{prof}_{\mathcal{O}_{\mathbb{P}_2,x}}(\nu_*\mathcal{O}_L)_x = \dim(\mathcal{O}_{\mathbb{P}_2,x}) - \operatorname{hd}_{\mathcal{O}_{\mathbb{P}_2,x}}(\nu_*\mathcal{O}_L)_x = 2-1 = 1$$
On the other hand: $\operatorname{prof}_{\mathcal{O}_{C,x}}(\nu_*\mathcal{O}_L)_x = \operatorname{prof}_{\mathcal{O}_{\mathbb{P}_2,x}}(\nu_*\mathcal{O}_L)_x = 1 = \dim(\nu_*\mathcal{O}_L)_x$.
- By (i) $\nu_*\mathcal{O}_L$ equals $\mathcal{O}_C$ on C^*. If (ii) is restricted to C we obtain the typical periodic resolution
$$\cdots \to \mathcal{O}_C(-4) \oplus \mathcal{O}_C(-3) \xrightarrow{A'} 2\mathcal{O}_C(-2) \xrightarrow{A} \mathcal{O}_C \oplus \mathcal{O}_C(-1) \to \nu_*\mathcal{O}_L \to 0$$
with
$$A' = \begin{pmatrix} -z_0 z_1 & z_2^2 \\ -z_2 & z_1 \end{pmatrix}$$
- The resolution (ii) is the "Beilinson resolution" of $\nu_*\mathcal{O}_L$. It is given by the complex
$$\mathcal{C}^k = \bigoplus_{k=i-j} H^i(\nu_*\mathcal{O}_L \otimes \Omega^j(j)) \otimes \mathcal{O}(-j)$$
and reduces to the non–zero terms C^{-1} and C^0 using the projection formula and $\nu^*\Omega^2(2) \cong \mathcal{O}_L(-3)$, $\nu^*\Omega^1(1) \cong \mathcal{O}_L(-2) \oplus \mathcal{O}_L(-1)$.

We denote by $\mathcal{D}_q$ the structure sheaf of the double point at $q = \langle 1, 0\rangle$ of L with resolution
$$0 \to \mathcal{O}_L(-2) \xrightarrow{t^2} \mathcal{O}_L \to \mathcal{D}_q \to 0.$$
It's easy to check that we have a decomposition
$$\nu^*\nu_*\mathcal{O}_L \cong \mathcal{O}_L \oplus \mathcal{D}_q.$$
Locally at p the sheaves $\nu_*\mathcal{O}_L(d)$ behave like $\nu_*\mathcal{O}_L$ because they are isomorphic there. Let's summarize some properties:

Lemma 4.2. (i) The sheaves $\nu_*\mathcal{O}_L(d)$ are Cohen-Macauley modules.
(ii) There is a canonical splitting $\nu^*\nu_*\mathcal{O}_L(d) \cong \mathcal{O}_L(d) \oplus \mathcal{D}_q$.
(iii) For any two integers a, b the canonical homorphism
$$\operatorname{Hom}(L, \mathcal{O}_L(a), \mathcal{O}_L(b)) \to \operatorname{Hom}(C, \nu_*\mathcal{O}_L(a), \nu_*\mathcal{O}_L(b))$$
is an isomorphism.
(iv) For any line bundle $\mathcal{L}$ on C we always have the canonical isomorphism
$$\operatorname{Hom}(C, \mathcal{L}, \nu_*\mathcal{O}_L(d)) \cong \operatorname{Hom}(L, \nu^*\mathcal{L}, \mathcal{O}_L(d)).$$

For any line bundle $\mathcal{L}$ on C the projection formula implies
$$(\nu_*\mathcal{O}_L) \otimes \mathcal{L} \cong \nu_*(\nu^*\mathcal{L}) = \nu_*\mathcal{O}_L(\deg \mathcal{L})$$
In particular $(\nu_*\mathcal{O}_L) \otimes \mathcal{L}_1 \cong (\nu_*\mathcal{O}_L) \otimes \mathcal{L}_2$ for any two line bundles of the same degree.

Definition 4.1. For any Cohen-Macaulay module $\mathcal{M}$ on C we define
$$\deg \mathcal{M} := \chi(\mathcal{M})$$

Is the again nothing but the Riemann–Roch formula $\chi(\mathcal{F}(m)) = r\deg(C)m + \deg(\mathcal{F}) + r\chi(\mathcal{O}_C)$ for coherent sheaves rank r sheaves on the cubic C. For $\nu_*\mathcal{O}_L(d)$ we have
$$\chi\nu_*\mathcal{O}_L(d) = \chi\mathcal{O}_L(d) = d+1$$
such that $\deg \nu_*\mathcal{O}_L(d) = d+1$.

We will find in 4.3. that $\nu_*\mathcal{O}_L$ has the same type of resolution as a degree 1 line bundle on C and can even be deformed into such a bundle. So the above definition is "justified".

Remark: We shall prove later that any true CM-module $\mathcal{M}$ on C is isomorphic to $\nu_*\mathcal{O}_L(d)$ where $d+1$ is the degree of $\mathcal{M}$. So the classification of these modules is the same as the classification of line bundles on L by integers. (cf. (iii) and (iv) of Lemma 4.2)

4.3. **Line bundles of degree 1 on C.** Let $\mathcal{F}$ be line bundle of degree 1 on C. Then $\nu^*\mathcal{F} \cong \mathcal{O}_L(1)$. By the Riemann–Roch theorem $\chi\mathcal{F} = \deg\mathcal{F} = 1$ and therefore $h^0\mathcal{F} > 0$. Any non–zero section of $\mathcal{F}$ defines a sequence

$$0 \to \mathcal{O}_C \to \mathcal{F} \to k_x \to 0$$

because the cokernel has length 1, since $\chi\mathcal{O}_C = 0$. Moreover the point x must be different from p because otherwise we would obtain the exact sequence $0 \to \mathcal{O}_L \to \mathcal{O}_L(1) \to \mathcal{D}_q \to 0$ if we applied ν^*. We have again that $\mathcal{F}$ is stable and that $P_{\mathcal{F}}(m) = 3m+1$. Thus, also the degree 1 line bundles lie in the component M_1 of $M_{3m+1}(\mathbb{P}_3)$. Due to Lemma 2.4, we obtain a resolution

$$0 \to 2\mathcal{O}(-2) \xrightarrow[\begin{pmatrix} q_1 & w_1 \\ q_2 & w_2 \end{pmatrix}]{} \mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{F} \to 0 \qquad (iii)$$

where q_1, q_2 are quadratic forms, w_1, w_2 are *independent* linear forms vanishing at x and $q_1w_2 - q_2w_1 = z_0z_1^2 - z_2^3$ is the equation of the cubic C. Up to the choice of the linear equations w_1, w_2 of x the matrix is unique.

Conversely, any point $x \in C^*$ defines via those matrices – up to isomorphism – a line bundle $\mathcal{F}$ of degree 1 with $h^0\mathcal{F} = 1$ and x is the zero locus of its "only" non-trivial section (cf. Lemma 2.4, (ii)⇒(i)).

Definition 4.2. For all $x \in C^*$ we denote by $\mathcal{O}_C(x)$ or $\mathcal{F}(x)$ the line bundle defined by a sequence (iii) where $w_1(x) = w_2(x) = 0$.

Then the isomorphism class $[\mathcal{O}_C(x)]$ is uniquely determined by x. We have obtained:

Lemma 4.3. The mapping $x \mapsto [\mathcal{O}_C(x)]$ defines a bijection $C \smallsetminus \{p\} = C^* \to \mathrm{Pic}_1(C)$.

Remarks:

- For all $x \in C^*$, $\mathcal{O}_C(x)$ is the line bundle associated to the divisor defined by x.
- By means of these resolutions (ii) and (iii) $\nu_*\mathcal{O}_L$ can be deformed into a bundle $\mathcal{O}_C(x)$ and is the unique degeneration for $x \to p$.

4.4. **The Picard group $\mathrm{Pic}(C)$.** Let x, y, z be different points on $C^* = C \smallsetminus \{p\}$. Then the bundle $\mathcal{O}_C(x) \otimes \mathcal{O}_C(y) \otimes \mathcal{O}_C(z)$ has a section s which vanishes exactly in $\{x, y, z\}$ and which is the tensor product of the unique sections of the single bundles.

Lemma 4.4. The following statements are equivalent

(i) $\mathcal{O}_C(x) \otimes \mathcal{O}_C(y) \otimes \mathcal{O}_C(z) \cong \mathcal{O}_C(1) = \mathcal{O}_C \otimes \mathcal{O}(1)$
(ii) $x + y + z = 0$ in C^*
(iii) x, y, z lie on a line

Corollary 4.1. For points $x, y, x', y' \in C^*$,

(i) $\mathcal{O}_C(x) \otimes \mathcal{O}_C(y) \cong \mathcal{O}_C(x') \otimes \mathcal{O}_C(y')$ if and only if $x + y = x' + y'$
(ii) $\mathcal{O}_C(x) \otimes \mathcal{O}_C(y) \cong \mathcal{O}_C(x+y) \otimes \mathcal{O}_C(0)$

Theorem 4.1. $\mathrm{Pic}(C) \cong \mathbb{Z} \times C^* \cong \mathbb{Z} \times (k, +)$, where $(k, +)$ is the additive group of the field k.

Remark: The map $Pic(C) \xrightarrow{\nu^*} Pic(L)$ is the projection $\mathbb{Z} \times C^* \to \mathbb{Z}$.

4.5. **The ideal sheaf $\mathcal{I}$ of the singular point p.** Let $\mathcal{I}$ be the ideal sheaf of the singular point with exact sequence $0 \to \mathcal{I} \to \mathcal{O}_C \to k_p \to 0$. It is generated by the two residue classes $\bar{z}_1, \bar{z}_2$ as sections of $\mathcal{O}_C(1)$. Their relations are generated by the rows of the matrix in the following sequence which is the resolution of $\mathcal{I}$ on $\mathbb{P}_2$.

$$0 \to \mathcal{O}(-2) \oplus \mathcal{O}(-3) \xrightarrow[\begin{pmatrix} -z_2 & z_1 \\ -z_0z_1 & z_2^2 \end{pmatrix}]{} 2\mathcal{O}(-1) \xrightarrow[\binom{\bar{z}_1}{\bar{z}_2}]{} \mathcal{I} \to 0 \qquad (iv)$$

It follows that $\mathcal{I}$ is not free at p and that $\mathcal{I}$ is a Cohen-Macaulay module. Dualizing this sequence we find that

$$\mathcal{I} \cong (\nu_*\mathcal{O}_L)^\vee \quad \text{or} \quad \mathcal{I}^\vee \cong \nu_*\mathcal{O}_L,$$

and we have $\deg\mathcal{I} = -1$. Moreover, the restriction to C of the resolution of $\nu_*\mathcal{O}_L$ shows that there is an exact sequence

$$0 \to \mathcal{I}(-1) \to \mathcal{O}_C \oplus \mathcal{O}_C(-1) \to \nu_*\mathcal{O}_L \to 0.$$

We shall see later that $\mathcal{I} \cong \nu_*\mathcal{O}_L(-2)$.

Lemma 4.5. $\mathcal{I}_p$, $(\nu_*\mathcal{O}_L)_p$ and all the kernels in the periodic resolution on C at p are isomorphic.

In particular all the modules $\mathcal{I}_p, (\nu_*\mathcal{O}_L(d))_p$ are isomorphic.

Now we investigate the homomorphisms $\mathcal{I} \to k_p$.

They factorize over $\mathcal{I}_p/\mathfrak{m}_p\mathcal{I}_p \cong \mathfrak{m}_p/\mathfrak{m}_p^2$ and thus are nothing but tangent vectors. They can also be described as homomorphisms $2\mathcal{O} \to k_p$ given by pairs $\binom{a}{b}$ of constants. Since a scalar multiple defines isomorphic kernels we may assume that $\binom{a}{b}$ is either

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$$

with $\alpha \in k$. In the first case the tangent vector defines the double tangent at p. Here we get the exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2\mathcal{O}(-3) & \xrightarrow{\begin{pmatrix} z_0z_1 & -z_2 \\ z_2^2 & -z_1 \end{pmatrix}} & \mathcal{O}(-1) \oplus \mathcal{O}(-2) & \longrightarrow & \mathcal{J} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle -\begin{pmatrix} 0 & 1 \\ z_2 & 0 \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & z_2 \end{pmatrix}} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-2) \oplus \mathcal{O}(-3) & \xrightarrow{\begin{pmatrix} -z_2 & z_1 \\ z_0z_1 & z_2^2 \end{pmatrix}} & 2\mathcal{O}(-1) & \longrightarrow & \mathcal{I} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_l(-2) & \xrightarrow{z_1} & \mathcal{O}_l(-1) & \longrightarrow & k_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}$$

where $l = \{z_2 = 0\}$. This shows that the kernel $\mathcal{J}$ is again a Cohen-Macaulay module and that $\mathcal{J}(1) \cong \nu_*\mathcal{O}_L$. In the second case we obtain the exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}(-1) & \xrightarrow{\overline{z}_2 - \alpha\overline{z}_1} & \mathcal{J}_\alpha & & \\
 & & & & \downarrow{\scriptstyle (-\alpha,1)} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-2) \oplus \mathcal{O}(-3) & \xrightarrow[\begin{pmatrix} -z_2 & z_1 \\ -z_0z_1 & z_2^2 \end{pmatrix}]{} & 2\mathcal{O}(-1) & \longrightarrow & \mathcal{I} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_\alpha = \begin{pmatrix} \alpha & -1 \\ -z_0 & \alpha z_2 \end{pmatrix}} & & \downarrow{\scriptstyle \binom{1}{\alpha}} & & \downarrow{\scriptstyle \pi_\alpha} & & \\
 & & 2\mathcal{O}(-2) & \xrightarrow[\binom{z_1}{z_2}]{} & \mathcal{O}(-1) & \longrightarrow & k_p & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

Since φ_α is an isomorphism at p, we find that $\overline{z}_2 - \alpha\overline{z}_1$ is a surjection $\mathcal{O}(-1) \to \mathcal{J}_\alpha$ at p. This shows that $\mathcal{J}_\alpha$ is a line bundle on C and that $\overline{z}_2 - \alpha\overline{z}_1$ vanishes exactly in the point

$$x(\alpha) = \langle \alpha^3, 1, \alpha \rangle,$$

which is the second intersection point with C of the line l_α through p which is defined by the tangent vector $\binom{1}{\alpha}$. We thus have the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{J}_\alpha(1) \to k_{x(\alpha)} \to 0$$

and $\mathcal{J}_\alpha(1)$ is the line bundle $\mathcal{J}_\alpha(1) \cong \mathcal{O}_C(x(\alpha))$. In case $\alpha = 0$ we get $\mathcal{J}_0(1) = \mathcal{O}_C(0)$.

Corollary 4.2. Let $\mathcal{L}_\pi$ be the kernel of the surjection $\nu_*\mathcal{O}_L(d) \xrightarrow{\pi} k_p$. Then $\mathcal{L}_\pi$ is a line bundle of degree d except for one π (corresponding to the double tangent at p) in which case $\mathcal{L}_\pi \cong \nu_*\mathcal{O}_L(d-1)$. In this case the sequence

$$0 \to \nu_*\mathcal{O}_L(d-1) \to \nu_*\mathcal{O}_L(d) \to k_p \to 0$$

is induced by the standard sequence

$$0 \to \mathcal{O}_L(-1) \to \mathcal{O}_L \to k_p \to 0.$$

4.6. **Classification of the global Cohen-Macaulay modules.** Now we have the tools to classify the true global Cohen-Macaulay modules:

Theorem 4.2. The global Cohen-Macauley modules on the cuspidal cubic C which are not free at the cusp p are classified by their degree, i.e. any true Cohen-Macaulay module $\mathcal{M}$ on C is isomorphic to $\nu_*\mathcal{O}_L(d)$ where $d+1 = \deg \mathcal{M} = \chi\mathcal{M}$.

Then we get immediately the following Corollary:

Corollary 4.3. (i) Let $\mathcal{M}_1, \mathcal{M}_2$ be two true Cohen-Macaulay modules on C. Then $\mathcal{M}_1 \cong \mathcal{M}_2$ if and only if $\nu^*\mathcal{M}_1 \cong \nu^*\mathcal{M}_2$.
(ii) $\mathcal{I} = \nu_*\mathcal{O}_L(-2)$ and $\mathcal{J} = \nu_*\mathcal{O}_L(-3)$.

Finally, obverve that:

Lemma 4.6. Any Cohen-Macaulay module on C is reflexive and we have the formula

$$(\nu_*\mathcal{O}_L(d))^\vee \cong \nu_*\mathcal{O}_L(-d-2).$$

The only module which is selfdual is $\nu_*\mathcal{O}_L(-1)$.

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