ON THE MODULI SCHEME OF STABLE SHEAVES SUPPORTED ON CUBIC SPACE CURVES

HANS GEORG FREIERMUTH AND GÜNTHER TRAUTMANN

Abstract. We investigate the geometry of the Simpson moduli space $M_P(\mathbb{P}_3)$ of stable sheaves with Hilbert polynomial $P(m) = 3m + 1$. It consists of two smooth, rational components M_0 and M_1 of dimensions 12 and 13 intersecting each other transversally along an 11-dimensional, smooth, rational subvariety. The component M_0 is isomorphic to the closure of the space of twisted cubics in the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}_3)$ and M_1 is isomorphic to the incidence variety of the relative Hilbert scheme of cubic curves contained in planes. In order to obtain the result and to classify the sheaves, we characterize $M_P(\mathbb{P}_3)$ as geometric quotient of a certain matrix parameter space by a non-reductive group. We also compute the Betti numbers of the Chow groups of the moduli space.

Contents

Date: April 11, 2002.

1991 *Mathematics Subject Classification.* Primary 14F05, 18F20; Secondary 14D20, 14D22.

1. Introduction

Generalizing the Maruyama schemes of semi-stable torsionfree coherent sheaves, C. Simpson introduced in 1994 coarse projective moduli spaces $M_P(X)$ for *arbitrary* semi-stable sheaves with fixed Hilbert polynomial P on a smooth projective variety X. If the degree d of the Hilbert polynomial P is strictly less than the dimension of X, all elements of the space $M_P(X)$ are sheaves supported on proper d-dimensional subvarieties of X.

One among many reasons for studying these Simpson moduli spaces is that they often contain structure sheaves of subvarieties of X with Hilbert polynomial P. Thus, certain components of $M_P(X)$ can be viewed as compactifications of an open part of the corresponding Hilbert scheme $\mathrm{Hilb}_P(X)$.

The structure of the spaces $M_P(\mathbb{P}_2)$ of semi-stable sheaves with linear polynomial $P(m) = \mu m + \chi$ and small multiplicities $\mu \leq 4$ supported on *plane curves* has been described by J. Le Potier [8].

In this paper, we investigate the geometry of the space $M_{3m+1}(\mathbb{P}_3)$, one of the first non-trivial examples of a reducible moduli space of stable sheaves on $\mathbb{P}_3$ supported on *space curves*.

One might view $M = M_{3m+1}(\mathbb{P}_3)$ in some sense as an analogue of the Hilbert scheme $\mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$ containing the twisted cubics, which has been analyzed by R. Piene and M. Schlessinger [11]. The two spaces share many similar properties. They consist for example both of two smooth rational components which intersect transversally. Indeed, even more is true: One component M_0 of the Simpson moduli space M is isomorphic to the closure H_0 of the space of twisted cubic curves in $\mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$. On the other hand, the second component $M_1 \subset M$ consisting of Cohen-Macaulay modules on planar cubics is simpler than the "ghost" component H_1 of the Hilbert scheme.

The structure sheaves of cubic space curves $\mathcal{O}_C \in M_0$ specialize in direction towards the intersection $M_0 \cap M_1$ in a natural way to rank-1 Cohen-Macaulay sheaves with support on *singular* plane cubics.

We give a complete classification of the sheaves in the moduli space M and of their resolutions. Furthermore, we characterize M as a geometric quotient of a parameter space X by a non-reductive algebraic group G. The points in X are the matrices that occur in a common resolution for all the sheaves in M. This representation as a quotient is used to compute an explicit deformation of the sheaves in $M_0 \cap M_1$ into structure sheaves of space cubics, to prove the transversality of the intersection $M_0 \cap M_1$, and to identify M_0 as a blow up. We summarize our result in the following theorem.

1.1. **Theorem.** *Let $M := M_{3m+1}(\mathbb{P}_3)$ be the Simpson moduli space of semi-stable sheaves on $\mathbb{P}_3$ with Hilbert polynomial $3m + 1$. Then*

(1) *M is a fine moduli space representing the corresponding functor.*

(2) *All sheaves in M are stable and admit a free resolution*

$$0 \longrightarrow 2\mathcal{O}_{\mathbb{P}_3}(-3) \xrightarrow{B} \mathcal{O}_{\mathbb{P}_3}(-1) \oplus 3\mathcal{O}_{\mathbb{P}_3}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{O}_{\mathbb{P}_3}(-1) \longrightarrow \mathcal{F} \longrightarrow 0 .$$

(3) *M is a projective variety and consists of two nonsingular, irreducible, rational components M_0 and M_1 of dimension* 12 *and* 13 *respectively.*

(4) *M_0 is isomorphic to the component H_0 of the Hilbert scheme* $\mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$ *which contains the twisted cubic curves. It is also isomorphic to the blow up of a space N of nets of quadrics in $\mathbb{P}_3$ along the subvariety N_1 of degenerate nets, see* [2] *and 8.2.*

(5) *The component M_1 consists of isomorphism classes of stable sheaves $\mathcal{F}$, supported on plane cubics C, which admit a non-split extension*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow k_p \longrightarrow 0 ,$$

where $p \in C$. M_1 is isomorphic to the incidence variety of the relative Hilbert scheme of cubic curves contained in planes. As a subscheme of M, the component M_1 is characterized by the vanishing of the degree 0 *entry of the matrices A in* (2).

(6) *The two components M_0 and M_1 intersect transversally and the intersection is smooth, rational of dimension* 11. *It is isomorphic to the exceptional divisor $\mathbb{P}(\mathcal{N}_{N_1/N})$ of the blow up $M_0 \cong Bl_{N_1}(N)$. The elements of $M_0 \cap M_1$ are Cohen-Macaulay sheaves supported on plane singular cubics with a section vanishing exactly in one of the singularities of the curve.*

(7) *There is a quasi–affine scheme X acted on by a non-reductive algebraic group G and a G–equivariant morphism $X \xrightarrow{\pi} M$ such that*
   (i) *the pullback to $X \times \mathbb{P}_3$ of the universal sheaf $\boldsymbol{\mathcal{U}}$ on $M \times \mathbb{P}_3$ has a universal resolution which restricts to the resolution of $\boldsymbol{\mathcal{U}}_{\pi(x)}$ of* (2) *at any x.*
   (ii) *$X \xrightarrow{\pi} M$ is a geometric quotient.*

The description of M_0 as a blow up of N is used in section 9 to compute the Chow groups of M based on a result of G. Ellingsrud and S.A. Strømme [2] on the Chow ring of the space of nets of quadrics, see theorem 9.2.

Acknowledgements. The first author wants to thank Michael Thaddeus for useful discussions and his encouragement. The second author was partially supported by the DFG–Schwerpunktprogramm “Globale Methoden in der komplexen Geometrie”.

Notation and Terminology. We work over a fixed, algebraically closed field k of characteristic 0 and denote projective n-space by $\mathbb{P} = \mathbb{P}_n = \mathbb{P}V$, where V is an $(n+1)$-dimensional vector space over k. We call any subscheme of a projective space over k a projective variety.

The word curve will mean a purely 1-dimensional projective scheme over k which is Cohen-Macaulay, i.e. its local rings are Cohen-Macaulay or its structure sheaf has no 0-dimensional torsion. A curve may be reducible and non-reduced.

For any (base-)scheme S of finite type over k and any projective variety X with very ample line bundle $\mathcal{O}_X(1)$, we denote the two projections from $S \times_k X$ to S and X by p and q respectively and use the following abbreviations for coherent sheaves $\boldsymbol{\mathcal{F}} \in \mathrm{Coh}(S \times_k X)$, $\mathcal{G} \in \mathrm{Coh}(S)$ and $\mathcal{H} \in \mathrm{Coh}(X)$.

$\mathcal{G} \boxtimes \mathcal{H} = p^*\mathcal{G} \otimes q^*\mathcal{H}$, $\widetilde{\mathcal{G}}(m) = \mathcal{G} \boxtimes \mathcal{O}_X(m) = p^*\mathcal{G} \otimes q^*\mathcal{O}_X(m)$, $\boldsymbol{\mathcal{F}}(m) = \boldsymbol{\mathcal{F}} \otimes p^*\mathcal{O}_X(m)$ and $\boldsymbol{\mathcal{F}}_s = \boldsymbol{\mathcal{F}}|_{X_s}$ for the fibers. If $f : S' \to S$ is a morphism we write f_X for $f \times id : S' \times_k X \to S \times_k X$. Usually, we will use bold letters for S-flat families $\boldsymbol{\mathcal{F}}$ on $S \times_k X$. Finally, we denote the support of $\mathcal{H} \in \mathrm{Coh}(X)$ defined by the annihilator ideal sheaf $\mathrm{Ann}(\mathcal{H})$ by $Z(\mathcal{H})$.

2. Schemes of stable sheaves

We recall the theorem of C. Simpson [12] on the existence of a coarse moduli scheme for semi-stable sheaves on a smooth projective variety X. A coherent $\mathcal{O}_X$–module is called **purely d–dimensional** if its support is purely d–dimensional and if it has no torsion in dimension $< d$. The Hilbert polynomial of a purely d–dimensional sheaf $\mathcal{F}$ can be written as

$$P_{\mathcal{F}}(m) := \sum_{\nu=0}^{d} (-1)^\nu \dim_k H^\nu(X, \mathcal{F} \otimes \mathcal{O}_X(m)) = \sum_{\nu=0}^{d} a_\nu(\mathcal{F}) \binom{m+\nu-1}{\nu} = \frac{a_d(\mathcal{F})}{d!} m^d + \dots$$

with integers $a_\nu(\mathcal{F})$. The number $\mu(\mathcal{F}) = a_d(\mathcal{F})$ is positive and is called the **multiplicity** of $\mathcal{F}$. The polynomial

$$R_{\mathcal{F}}(m) = P_{\mathcal{F}}(m)/a_d(\mathcal{F})$$

is called the **reduced Hilbert polynomial** of $\mathcal{F}$. We write $R_{\mathcal{E}} \leq R_{\mathcal{F}}$ resp. $R_{\mathcal{E}} < R_{\mathcal{F}}$ if $R_{\mathcal{E}}(m) \leq R_{\mathcal{F}}(m)$ resp. $R_{\mathcal{E}}(m) < R_{\mathcal{F}}(m)$ for large m. A coherent sheaf $\mathcal{F}$ of pure dimension d is called **semi-stable** resp. **stable** if for any proper subsheaf $0 \neq \mathcal{E} \subsetneq \mathcal{F}$,

$$R_{\mathcal{E}} \leq R_{\mathcal{F}} \quad \text{resp.} \quad R_{\mathcal{E}} < R_{\mathcal{F}}.$$

It is easy to prove that $\mathcal{F}$ is (semi)-stable if and only if for any proper quotient sheaf $\mathcal{G}$ of $\mathcal{F}$ which is also purely d–dimensional,

$$R_{\mathcal{F}} \leq R_{\mathcal{G}} \quad \text{resp.} \quad R_{\mathcal{F}} < R_{\mathcal{G}},$$

see [6], proposition 1.2.6.

Given a numerical polynomial $P \in \mathbb{Q}[T]$, a contravariant functor

$$\mathcal{M}_P(X) : (\text{schemes}) \to (\text{sets})$$

is defined as follows. For a scheme S, the set $\mathcal{M}_P(X)(S)$ is the set of all classes $[\boldsymbol{\mathcal{F}}]$ of coherent sheaves $\boldsymbol{\mathcal{F}}$ on $S \times X$ which are S–flat and for which all fibers $\mathcal{F}_s$, $s \in S$, are semi-stable with Hilbert polynomial $P_{\mathcal{F}_s}(m) = P(m)$. Here the class $[\boldsymbol{\mathcal{F}}]$ is defined to be the class of $\boldsymbol{\mathcal{F}}$ under the equivalence relation $\boldsymbol{\mathcal{F}} \sim \boldsymbol{\mathcal{F}} \otimes p^*\mathcal{L}$ for a line bundle $\mathcal{L}$ on S. For a morphism $T \xrightarrow{f} S$ the map

$$\mathcal{M}_P(X)(S) \to \mathcal{M}_P(X)(T)$$

is defined by $[\boldsymbol{\mathcal{F}}] \mapsto [f_X^*\boldsymbol{\mathcal{F}}]$.

2.1. **Theorem** (Simpson). *Let X be a smooth projective variety and P a numerical polynomial. Then there exists a projective variety $M_P(X)$ which is a coarse moduli space corepresenting the functor $\mathcal{M}_P(X)$.*
The closed points of $M_P(X)$ parametrize the S-equivalence classes of semi-stable sheaves on X with fixed Hilbert polynomial P.
Furthermore, there is an open subset $M_P^s(X) \subset M_P(X)$ parametrizing the isomorphism classes of stable sheaves.

For proofs, see [12],[7],[6]. If the integers $a_\nu(\mathcal{F})$ in the Hilbert polynomial are pairwise coprime, then any semi-stable sheaf is already stable and thus $M_P(X) = M_P^s(X)$. Moreover, in this case there exists a universal sheaf $\boldsymbol{\mathcal{U}}$ on $M_P(X) \times X$ such that $M_P(X)$ becomes a fine moduli space and represents the functor $\mathcal{M}_P(X)$, see [6], corollary 4.6.6. There is also a relative version of Simpson's theorem, see [6], theorem 4.3.7.

2.2. **Proposition.** *The tangent space at a stable sheaf $[\mathcal{F}]$ in $M_P^s(X)$ is isomorphic to $Ext^1_{\mathcal{O}_X}(X, \mathcal{F}, \mathcal{F})$ and the germ of $M_P^s(X)$ at $[\mathcal{F}]$ is a universal deformation of $\mathcal{F}$. If $Ext^2_{\mathcal{O}_X}(X, \mathcal{F}, \mathcal{F}) = 0$, then $M_P^s(X)$ is smooth at $[\mathcal{F}]$.*

A proof can be found in [6], corollary 4.5.2. For the polynomial $P(m) = 3m+1$ we obtain in particular:

2.3. **Proposition.** *a) Any semi-stable sheaf on $\mathbb{P}_3$ with Hilbert polynomial $3m+1$ is stable and purely 1-dimensional, and $M = M_{3m+1}(\mathbb{P}_3) = M_{3m+1}^s(\mathbb{P}_3)$.*
b) M represents the functor $\mathcal{M}_{3m+1}(\mathbb{P}_3)$ and there is a universal sheaf $\boldsymbol{\mathcal{U}}$ over $M \times \mathbb{P}_3$.

3. Stable sheaves supported on cubics

The Hilbert Scheme $H := \mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$ consists of two smooth, rational components. One of them is the 12-dimensional Zariski closure H_0 of the space of twisted cubic curves in H. The other 15-dimensional component H_1 contains the planar cubics with an additional point in $\mathbb{P}_3$. Piene and Schlessinger showed in [11] that $H_0 \cap H_1$ is smooth, rational of dimension 11 and that the two components intersect transversally. The curves in the intersection are planar, singular cubics with an embedded point at one of the singularities.

Now we determine the types of the sheaves in M and their resolutions.

We recall first a few facts about curves C in $\mathbb{P}_3$ of degree d:

(C1) The arithmetic genus satisfies $p_a(C) \leq \binom{d-1}{2}$ with equality if and only if C is contained in a plane $H \subset \mathbb{P}_3$.
(C2) If C is not planar then $h^1(\mathcal{I}_C(n)) \leq \binom{d-2}{2} - p_a(C)$ for all $n \in \mathbb{Z}$.
(C3) $\operatorname{reg}(\mathcal{I}_C) \leq \binom{d}{2} + 1 - p_a(C)$, where reg denotes the Mumford-Castelnuovo regularity.

3.1. **Lemma.** *For any cubic curve C in $H_0 \setminus H_1$ the structure sheaf $\mathcal{O}_C$ is stable.*

Proof. The curves in $H_0 \setminus H_1$ are precisely the connected Cohen-Macaulay curves of degree 3 in $\mathbb{P}_3$ which are not contained in a plane and have Hilbert polynomial $P_{\mathcal{O}_C}(m) = 3m+1$, see e.g. [11] or [5]. Given a proper quotient $\mathcal{O}_C \to \mathcal{Q}$ of pure dimension 1, $\mathcal{Q}$ is the structure sheaf $\mathcal{O}_{C'}$ of a (Cohen-Macaulay) curve $C' \subset C$ of $\deg(C') \leq 2$. If $\deg(C') = 1$, then C' is a line with $R_{\mathcal{O}_{C'}}(m) = m+1$ and thus $R_{\mathcal{O}_C} < R_{\mathcal{Q}}$. If $\deg(C') = 2$, then C' is a union of two skew lines or a conic and thus $P_{\mathcal{O}_{C'}}(m) = 2m+2$ or $2m+1$. In both cases $R_{\mathcal{O}_C} < R_{\mathcal{Q}}$. □

3.2. **Lemma.** *Let $\mathcal{F}$ be a stable sheaf on $\mathbb{P}_3$ with Hilbert polynomial $3m+1$. Then the support $C = Z(\mathcal{F})$ of $\mathcal{F}$ is a connected cubic curve of degree 3.*

(i) *If C is contained in a plane $H \subset \mathbb{P}_3$ then there exists a non-split extension*

$$0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{F} \longrightarrow k_p \to 0,$$

where $p \in C$ and k_p denotes the skyscraper sheaf supported on that point.

(ii) *If C is not contained in a plane, then $C \in H_0 \setminus H_1 \subset \mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$ and $\mathcal{F}$ is isomorphic to the structure sheaf $\mathcal{O}_C$.*

Proof. Because $P_{\mathcal{F}}(0) = 1$, the sheaf $\mathcal{F}$ has a non-zero section s. Let C denote the support of s. The corresponding homomorphism $\mathcal{O} \to \mathcal{F}$ induces an embedding $\mathcal{O}_C \subset \mathcal{F}$. Since $\mathcal{F}$ is purely 1-dimensional, also C is purely 1-dimensional and Cohen-Macaulay or a curve. We have $P_{\mathcal{O}_C}(m) \leq P_{\mathcal{F}}(m)$ for large m and thus $1 \leq \deg(C) \leq 3$. If $\deg(C) \leq 2$, then $P_{\mathcal{O}_C}(m) = m+1$, $2m+1$ or $2m+2$, depending on whether C is a line, a conic or a union of two skew lines. But this is excluded by the stability of $\mathcal{F}$. Therefore, C is a cubic curve. It is connected, because any proper connected component of it would violate stability, too.

(i) If C is contained in a plane, then $P_{\mathcal{O}_C}(m) = 3m$. It follows that the cokernel $\mathcal{F}/\mathcal{O}_C$ is a skyscraper sheaf k_p. Because $\mathcal{F}$ is stable, it cannot be isomorphic to the direct sum $\mathcal{O}_C \oplus k_p$. This proves (i). Furthermore, the ideal sheaf $\mathcal{I}_C$ also annihilates $\mathcal{F}$, because, if g is germ in $\mathcal{I}_C$, then $g\mathcal{F}$ has 0-dimensional support and must vanish because $\mathcal{F}$ has no 0-dimensional torsion. This proves that $C = Z(\mathcal{F})$ in the planar case.

(ii) Assume now that $Z(\mathcal{F})$ is not contained in a plane. Then also C is not contained in a plane by the previous part (i). Then $1 - P_{\mathcal{O}_C}(0) = p_a(C) < 1$ or $0 < P_{\mathcal{O}_C}(0)$, see (C1). On the other hand, the stability of $\mathcal{F}$ implies $P_{\mathcal{O}_C}(0)/3 \leq 1/3$ and hence $P_{\mathcal{O}_C}(0) = 1$. Therefore $P_{\mathcal{O}_C}(m) = 3m+1 = P_{\mathcal{F}}(m)$, which yields $\mathcal{O}_C = \mathcal{F}$. Moreover, $C = Z(\mathcal{F})$ must be contained in $H_0 \setminus H_1$. This completes the proof. Note, that in both cases $h^0(\mathcal{F}) = 1$. □

3.3. **Lemma.** *Let $\mathcal{F}$ be a stable sheaf on $\mathbb{P}_3$ with Hilbert polynomial $3m+1$ and let $C = Z(\mathcal{F})$ be its supporting cubic curve. Then the following are equivalent:*

(i) *There exists a non-split extension of type $0 \to \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow k_p \to 0$ with $p \in C$.*
(ii) *$\mathcal{F}$ is supported on a plane H and has a free resolution*

$$0 \to 2\mathcal{O}_H(-2) \underset{\left(\begin{smallmatrix} q_1 & l_1 \\ q_2 & l_2 \end{smallmatrix}\right)}{\longrightarrow} \mathcal{O}_H \oplus \mathcal{O}_H(-1) \to \mathcal{F} \to 0, \tag{1}$$

Remark: It can easily be verified that the resolution (1) is the Beilinson resolution of $\mathcal{F}$ on the plane H.

Proof. (i) to (ii): The extension sequence implies that $P_{\mathcal{O}_C}(m) = 3m$ and hence $p_a(C) = 1$. Then C is contained in a plane H. Because C is the support of $\mathcal{F}$, the extension sequence is a sequence of $\mathcal{O}_H$-modules. By standard homological algebra, a resolution of $\mathcal{F}$ can be obtained from the Koszul resolution of k_p twisted by $\mathcal{O}_H(-1)$ and the resolution of $\mathcal{O}_C$ by its cubic form f. The result is the resolution

$$0 \to \mathcal{O}_H(-3) \xrightarrow{B} \mathcal{O}_H(-3) \oplus 2\mathcal{O}_H(-2) \xrightarrow{A} \mathcal{O}_H \oplus \mathcal{O}_H(-1) \to \mathcal{F} \to 0$$

with

$$A = \begin{pmatrix} f & 0 \\ q_1 & l_1 \\ q_2 & l_2 \end{pmatrix} \quad \text{and} \quad B = (\lambda, -l_2, l_1),$$

such that in particular $\lambda f = q_1\, l_2 - q_2\, l_1$. Suppose $\lambda = 0$. Then $l_1 | q_1$ and $l_2 | q_2$ since l_1 and l_2 are independent. Hence there exists a linear form w such that $q_1 = w\, l_1$ and $q_2 = w\, l_2$, and it would follow that A is equivalent to $\begin{pmatrix} f & 0 \\ 0 & l_1 \\ 0 & l_2 \end{pmatrix}$. But then $\mathcal{F}$ decomposes into $\mathcal{F} \cong \mathcal{O}_C \oplus k_p$ contradicting the stability of $\mathcal{F}$. Consequently, we can assume $\lambda = 1$. Then

$$A = \begin{pmatrix} q_1\, l_2 - q_2\, l_1 & 0 \\ q_1 & l_1 \\ q_2 & l_2 \end{pmatrix} \quad \text{is equivalent to} \quad \begin{pmatrix} 0 & 0 \\ q_1 & l_1 \\ q_2 & l_2 \end{pmatrix},$$

and thus one can delete the ghost summand $\mathcal{O}_H(-3)$. The homomorphism of the matrix $\begin{pmatrix} q_1 & l_1 \\ q_2 & l_2 \end{pmatrix}$ is injective because $q_1\, l_2 - q_2\, l_1 \neq 0$.

(ii) to (i): Since the cokernel of the composition of the projection $\mathcal{O}_H \oplus \mathcal{O}_H(-1) \to \mathcal{O}_H(-1)$ with the matrix of the resolution (1) is k_p, we obtain a surjective map $\mathcal{F} \to k_p$. It follows easily that its kernel is $\mathcal{O}_C$.

□

3.4. **Lemma.** *Let C be a curve in $H_0 \smallsetminus H_1 \subset \mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$. Then $\mathcal{O}_C$ has a free resolution*

$$0 \longrightarrow 2\mathcal{O}_{\mathbb{P}_3}(-3) \longrightarrow 3\mathcal{O}_{\mathbb{P}_3}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}_3} \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{2}$$

Proof. Since C is not contained in a plane we get $h^0\mathcal{I}_C(1) = 0$. The ideal sheaf $\mathcal{I}_C$ is 4-regular and $h^1\mathcal{I}_C(2)$ vanishes, cf. remarks (C1) – (C3). Hence $P_{\mathcal{I}_C}(m) = \frac{1}{6}m^3 + m^2 - \frac{7}{6}m$ is equal to $h^0\mathcal{I}_C(m)$ for all $m \geq 2$. This implies that the saturated homogeneous ideal

$I = H^0_*(\mathcal{I}_C)$ of the curve is minimally generated by three independent quadratic forms $I = (q_1, q_2, q_3)$. Since the structure sheaf $\mathcal{O}_C$ is Cohen-Macaulay, we obtain an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow 3\mathcal{O}_{\mathbb{P}_3} \xrightarrow[\phi = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}]{} \mathcal{I}_C(2) \longrightarrow 0,$$

where $\mathcal{E}$ is a locally free sheaf of rank 2 on $\mathbb{P}_3$. $H^1_*(\mathcal{E}) = 0$ because the map ϕ is already surjective on global sections. Furthermore, (C2) implies $H^2\mathcal{E}(m) \cong H^1\mathcal{I}_C(2+m) = 0$ for all $m \in \mathbb{Z}$. Thus, due to Horrock's criterion, see [10], the vanishing of the intermediate cohomology implies that the bundle $\mathcal{E}$ splits, i.e. $\mathcal{E} = \mathcal{O}_{\mathbb{P}_3}(a) \oplus \mathcal{O}_{\mathbb{P}_3}(b)$. We know that $a, b < 0$. Using Riemann-Roch, we get $c_1(\mathcal{I}_C(2)) = 2$ since $\mathrm{rk}(\mathcal{I}_C(2)) = 1$ and $P_{\mathcal{I}_C(2)}(m) = \frac{1}{6}m^3 + 2m^2 + \frac{29}{6}m + 3$. It follows that $a = b = -1$ because $0 = c_1(\mathcal{E}) + c_1(\mathcal{I}_C(2)) = a + b + 2$. $\square$

3.5. **Proposition.** *Every stable sheaf $\mathcal{F}$ on $\mathbb{P}_3$ with $P_{\mathcal{F}}(m) = 3m+1$ has a free resolution*

$$0 \to 2\,\mathcal{O}_{\mathbb{P}_3}(-3) \xrightarrow{B} \mathcal{O}_{\mathbb{P}_3}(-1) \oplus 3\,\mathcal{O}_{\mathbb{P}_3}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{O}_{\mathbb{P}_3}(-1) \to \mathcal{F} \to 0. \qquad (CR)$$

If $C = Z(\mathcal{F})$ is contained in a plane $H = Z(w)$ the matrices can be chosen in normal form

$$B = \begin{pmatrix} -q_1 & -l_1 & w & 0 \\ -q_2 & -l_2 & 0 & w \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} w & 0 \\ 0 & w \\ q_1 & l_1 \\ q_2 & l_2 \end{pmatrix}, \textit{ where } I_C = (q_1\, l_2 - q_2\, l_1,\ w) \textit{ and } l_1 \wedge l_2 \neq 0.$$

If $Z(\mathcal{F})$ is not contained in any plane then the normal forms are

$$B = \begin{pmatrix} 0 & l_1 & l_2 & l_3 \\ 0 & l_4 & l_5 & l_6 \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} 0 & 1 \\ q_1 & 0 \\ q_2 & 0 \\ q_3 & 0 \end{pmatrix}$$

with linear forms l_i and quadratic forms q_i.

Proof. Say, $Z(\mathcal{F}) \subset H$. Then by lemma 3.2.(i) and lemma 3.3.(ii), there exists a resolution

$$0 \to 2\mathcal{O}_H(-2) \xrightarrow[\left(\begin{smallmatrix} q_1 & l_1 \\ q_2 & l_2 \end{smallmatrix}\right)]{} \mathcal{O}_H \oplus \mathcal{O}_H(-1) \to \mathcal{F} \to 0$$

of $\mathcal{F}$. Since the equation w of the plane H is annihilated by $\mathcal{F}$ also the map $\mathcal{O}_{\mathbb{P}_3}(-1) \oplus 3\,\mathcal{O}_{\mathbb{P}_3}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{O}_{\mathbb{P}_3}(-1)$ has $\mathcal{F}$ as cokernel. It is then easy to verify that the matrix of relations for A is indeed B. The non-planar case follows immediately from lemma 3.2.(ii) and lemma 3.4. $\square$

Remark: From the resolution (CR) in 3.5 we see that the Castelnuovo-Mumford regularity of any stable sheaf with Hilbert polynomial $3m+1$ is equal to 1 and hence $H^1\mathcal{F} = 0$. This fact and the Beilinson-II spectral sequence

$$H^s(\mathbb{P}_3, \mathcal{F} \otimes \Omega^{-r}_{\mathbb{P}_3}(-r)) \otimes_k \mathcal{O}_{\mathbb{P}_3}(r) =: E_1^{rs} \Longrightarrow E_\infty^i \cong \begin{cases} \mathrm{gr}(\mathcal{F}), & \text{for } i = 0 \\ 0, & \text{otherwise} \end{cases}$$

induce an exact sequence

$$\begin{aligned} 0 \to H^1\mathcal{F}(-1) \otimes \mathcal{O}_{\mathbb{P}_3}(-3) \to H^0(\mathcal{F} \otimes \Omega^1_{\mathbb{P}_3}(1)) \otimes \mathcal{O}_{\mathbb{P}_3}(-1) \oplus H^0(\mathcal{F} \otimes \Omega^2_{\mathbb{P}_3}(2)) \otimes \mathcal{O}_{\mathbb{P}_3}(-2) \\ \to H^0\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_3} \oplus H^1(\mathcal{F} \otimes \Omega^1_{\mathbb{P}_3}(1)) \otimes \mathcal{O}_{\mathbb{P}_3}(-1) \to \mathcal{F} \to 0 \end{aligned} \tag{3}$$

because the cohomology groups $H^0\mathcal{F}(-j),\ j > 0$, vanish due to the stability of $\mathcal{F}$.

Using the two sequences (1) and (2), it is an easy exercise in cohomology to check that

$$\begin{aligned} &h^0(\mathcal{F} \otimes \Omega^3(3)) = 0 \quad , \quad h^1(\mathcal{F} \otimes \Omega^3(3)) = 2 \\ &h^0(\mathcal{F} \otimes \Omega^2(2)) = 0 \quad , \quad h^1(\mathcal{F} \otimes \Omega^2(2)) = 3 \\ &h^0(\mathcal{F} \otimes \Omega^1(1)) = h^1(\mathcal{F} \otimes \Omega^1(1)) = \begin{cases} 1 & \text{if } Z(\mathcal{F}) \text{ is planar.} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, the resolution (CR) is the Beilinson resolution (3) if the sheaf is supported on a plane. In the case of non-planar sheaves, the terms $\mathcal{O}_{\mathbb{P}_3}(-1)$ can be cancelled from (CR) to give the Beilinson resolution (3) in that case. Furthermore, due to the explicit construction in this section we also know *exactly which types of matrices A and B can occur.*

The relative version of Beilinson's theorem, see [10], p. 306, specializes in our case to

3.6. **Proposition.** *1) Let $\boldsymbol{\mathcal{F}} \in Coh(S \times \mathbb{P}_3)$ be an S-flat family of semi-stable sheaves $\boldsymbol{\mathcal{F}}_s$ with Hilbert polynomial $P_{\boldsymbol{\mathcal{F}}_s}(m) = 3\,m + 1$ for all $s \in S$. Then there is an exact sequence*

$$\begin{aligned} 0 \to \mathcal{B}_2 \boxtimes \mathcal{O}_{\mathbb{P}_3}(-3) \to \mathcal{A}_1 \boxtimes \mathcal{O}_{\mathbb{P}_3}(-1) \oplus \mathcal{B}_1 \boxtimes \mathcal{O}_{\mathbb{P}_3}(-2) \\ \to \mathcal{A}_0 \boxtimes \mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{B}_0 \boxtimes \mathcal{O}_{\mathbb{P}_3}(-1) \to \boldsymbol{\mathcal{F}} \to 0 \end{aligned}$$

with

$$\begin{aligned} & & \mathcal{B}_2 &= R^1p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^3(3)) \\ \mathcal{A}_1 &= p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1(1)) & \mathcal{B}_1 &= R^1p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^2(2)) \\ \mathcal{A}_0 &= p_*\boldsymbol{\mathcal{F}} & \mathcal{B}_0 &= R^1p_*(\boldsymbol{\mathcal{F}} \otimes \Omega^1(1)) \end{aligned}$$

2) The sheaves $\mathcal{B}_2, \mathcal{B}_1, \mathcal{A}_0$ are locally free of rank $2, 3, 1$ *respectively.*

3) The sheaves $\mathcal{A}_1$ and $\mathcal{B}_0$ are supported on the subscheme $S_1 \subset S$ of points $s \in S$ for which $\boldsymbol{\mathcal{F}}_s$ is planar.

Proof. The first part follows from the relative Beilinson spectral sequence and the vanishing of cohomology groups $H^0(\boldsymbol{\mathcal{F}}_s(-j))$ of the stable sheaves $\boldsymbol{\mathcal{F}}_s, s \in S$. The second part follows from the base change theorem for the points of S using that $h^1(\boldsymbol{\mathcal{F}}_s \otimes \Omega^3(3)) = 2$, $h^1(\boldsymbol{\mathcal{F}}_s \otimes \Omega^2(2)) = 3$ and $h^0(\boldsymbol{\mathcal{F}}_s) = 1$ for any s and the vanishing $h^0(\boldsymbol{\mathcal{F}}_s \otimes \Omega^3(3)) = h^0(\boldsymbol{\mathcal{F}}_s \otimes \Omega^2(2)) = 0$ (the scheme S might be non–reduced).

Because $\boldsymbol{\mathcal{F}}$ is S–flat, the function $s \mapsto h^0(\boldsymbol{\mathcal{F}}_s \otimes \Omega^1(1)) = h^1(\boldsymbol{\mathcal{F}}_s \otimes \Omega^1(1))$ is upper semi-continuous and defines a reduced, closed subscheme $S_1 \subset S$ as its support. Then $S_1 = Z(\mathcal{A}_1) = Z(\mathcal{B}_0)$. □

There arise two non–standard problems from the last two propositions: The resolution of type (CR) is not the Beilinson resolution if the sheaf is the structure sheaf $\mathcal{O}_C$ of a non-planar cubic. Therefore, the Beilinson construction cannot be used directly in order to describe the parameters for the moduli space. Secondly, the automorphism group of a complex (CR) is not reductive. Both problems are overcome in section 5.

4. The two components

Let $M = M_{3m+1}(\mathbb{P}_3)$. As stated in proposition 2.3, there is a universal sheaf $\boldsymbol{\mathcal{U}}$ on $M \times \mathbb{P}_3$. By proposition 3.6, 3), there exists a closed subscheme $M_1 \subset M$ with reduced structure consisting of all sheaves with planar support. We will show in section 6 that M_1 is irreducible, smooth and rational of dimension 13. Let now M_0 denote the closure of the open subscheme $M \smallsetminus M_1$ of non–planar sheaves. Then we have

$$M = M_0 \cup M_1.$$

4.1. **Lemma.** *$M \smallsetminus M_1$ is isomorphic to $H_0 \smallsetminus H_1$, the open part of non–planar cubics of the Hilbert scheme $Hilb_{3m+1}(\mathbb{P}_3)$.*

Proof. There is a set-theoretical bijection $\phi : M_0 \smallsetminus M_1 \longrightarrow H_0 \smallsetminus H_1$ due to lemma 3.2, (ii). This bijection is indeed an isomorphism: Let $Z_0 \subset (H_0 \smallsetminus H_1) \times \mathbb{P}_3$ be the universal cubic of $\mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$ restricted to $H_0 \smallsetminus H_1$. Then there is a resolution

$$0 \to \mathcal{E}_1 \boxtimes \mathcal{O}_{\mathbb{P}_3}(-3) \to \mathcal{E}_0 \boxtimes \mathcal{O}_{\mathbb{P}_3}(-2) \to \mathcal{O}_{(H_0 \smallsetminus H_1) \times \mathbb{P}_3} \longrightarrow \mathcal{O}_{Z_0} \to 0$$

with $\mathcal{E}_1$ and $\mathcal{E}_0$ locally free on $H_1 \smallsetminus H_0$ of rank 2 and 3. It restricts to the resolution from lemma 3.4 along the fibers. On the other hand, let $\boldsymbol{\mathcal{U}}_0$ denote the restriction of the universal sheaf $\boldsymbol{\mathcal{U}}$ to $(M \smallsetminus M_1) \times \mathbb{P}_3$. Since the sheaves $\mathcal{A}_1$ and $\mathcal{B}_0$ in the resolution of $\boldsymbol{\mathcal{U}}$ vanish on $M \smallsetminus M_1$ (cf. proposition 3.6), we get

$$0 \to (\mathcal{B}_2 \otimes \mathcal{A}_0^*) \boxtimes \mathcal{O}_{\mathbb{P}_3}(-3) \to (\mathcal{B}_1 \otimes \mathcal{A}_0^*) \boxtimes \mathcal{O}_{\mathbb{P}_3}(-2) \to \mathcal{O}_{(M \smallsetminus M_1) \times \mathbb{P}_3} \longrightarrow \boldsymbol{\mathcal{U}}_0 \otimes p^* \mathcal{A}_0^* \to 0.$$

Because $\boldsymbol{\mathcal{U}}_0 \otimes p^*\mathcal{A}_0^*$ represents a family of cubic curves in $H_0 \smallsetminus H_1$, it follows from the universal property of H and its open subset $H_0 \smallsetminus H_1$, that there is a unique morphism $M_0 \smallsetminus M_1 \xrightarrow{\alpha} H_0 \smallsetminus H_1$ such that $\boldsymbol{\mathcal{U}}_0 \otimes p^*\mathcal{A}_0^*$ is the pullback of $\mathcal{O}_{Z_0}$. Since on the other hand $\mathcal{O}_{Z_0}$ is also a family of stable sheaves belonging to $M_0 \smallsetminus M_1$, there is a unique morphism $H_0 \smallsetminus H_1 \xrightarrow{\beta} M_0 \smallsetminus M_1$ such that conversely $\mathcal{O}_{Z_0}$ is the pullback of $\boldsymbol{\mathcal{U}}_0 \otimes p^*\mathcal{A}_0^*$. By uniqueness α and β are inverse to each other. The underlying map of α is exactly the bijection ϕ above. □

Piene and Schlessinger's result [11] on $H_0 \subset \mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$ implies then the following corollary.

4.2. **Corollary.** *The subscheme M_0 is an irreducible component of M of dimension 12 and smooth along $M_0 \smallsetminus M_1$.*

We will show in section 8 that the component M_0 as a whole is isomorphic to H_0 and that it can be identified with a blowup of the space of nets of quadrics as described in [2] such that $M_0 \cap M_1$ is the exceptional divisor.

5. A parameter space

The Beilinson resolution of proposition 3.6 over $M \times \mathbb{P}_3$ of the universal sheaf contains the sheaves $\mathcal{A}_1$ and $\mathcal{B}_0$ which are supported on M_1 and so are not locally free on M. On the other hand, they do not cancel because the homomorphism between them is zero. In the following, we construct a quasi–affine variety X consisting of the matrices which occur in the resolution in proposition 3.5 such that the moduli space M is a geometric quotient of X and such that the variety X is the minimal one allowing a locally free resolution of the lifted universal sheaf. To begin with, we prove the following

5.1. **Lemma.** *Any $\mathcal{F}$ in M has an open neighbourhood $U \subset M$ such that there is a resolution*

$$0 \to 2\widetilde{\mathcal{O}}_U(-3) \to \widetilde{\mathcal{O}}_U(-1) \oplus 3\widetilde{\mathcal{O}}_U(-2) \to \widetilde{\mathcal{O}}_U \oplus \widetilde{\mathcal{O}}_U(-1) \to \boldsymbol{\mathcal{U}}|_{U\times\mathbb{P}_3} \to 0,$$

where $\widetilde{\mathcal{O}}_U(d) = \mathcal{O}_U \boxtimes \mathcal{O}_{\mathbb{P}_3}(d)$ and $\boldsymbol{\mathcal{U}}$ denotes the universal sheaf on $M \times \mathbb{P}_3$.

Proof. Consider the relative Beilinson resolution of $\boldsymbol{\mathcal{U}}$ from proposition 3.6. If $\mathcal{F} \in M_0 \smallsetminus M_1$ choose an open neighbourhood $U \subset M_0 \smallsetminus M_1$ of $\mathcal{F}$ and add the complex $0 \to \widetilde{\mathcal{O}}_U(-1) = \widetilde{\mathcal{O}}_U(-1) \to 0$ to the restriction of the resolution to $U \times \mathbb{P}_3$. In the case $\mathcal{F} \in M_1 \smallsetminus M_0$, the restriction of the resolution to some open neighbourhood $\mathcal{F} \in U \subset M_1 \smallsetminus M_0$ is already of the proposed type since $\mathcal{A}_1$ and $\mathcal{B}_0$ are locally free on their support M_1.

So let $\mathcal{F} \in M_0 \cap M_1$. There exists an open neighbourhood $U_1 \subset M_1$ of $\mathcal{F}$ with

$$0 \to 2\widetilde{\mathcal{O}}_{U_1}(-3) \xrightarrow{\widetilde{B}_1} \widetilde{\mathcal{O}}_{U_1}(-1) \oplus 3\widetilde{\mathcal{O}}_{U_1}(-2) \xrightarrow{\widetilde{A}_1} \widetilde{\mathcal{O}}_{U_1} \oplus \widetilde{\mathcal{O}}_{U_1}(-1) \xrightarrow{\widetilde{\varphi}_1} \boldsymbol{\mathcal{U}}|_{U_1\times\mathbb{P}_3} \to 0\ .$$

Here the component of $\widetilde{A}_1$ from $\widetilde{\mathcal{O}}_{U_1}(-1)$ to $\widetilde{\mathcal{O}}_{U_1}(-1)$ is zero. We are going to extend this resolution to a resolution of $\boldsymbol{\mathcal{U}}$ on an open set U in M with $U \cap M_1 = U_1$.

For that let $\mathcal{I}$ denote the ideal sheaf of $M_1 \subset M$. The exact sequence $0 \to p^*\mathcal{I} \otimes \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{U}}|_{M_1\times\mathbb{P}_3} \to 0$ induces

$$H^0(U \times \mathbb{P}_3, \boldsymbol{\mathcal{U}}(i)) \longrightarrow H^0(U_1 \times \mathbb{P}_3, \boldsymbol{\mathcal{U}}(i)|_{M_1\times\mathbb{P}_3}) \longrightarrow H^1(U \times \mathbb{P}_3, p^*\mathcal{I} \otimes \boldsymbol{\mathcal{U}}(i)), \quad i = 0, 1$$

where $U \subset M$ is some open affine with $U \cap (M_0 \smallsetminus M_1) \neq \emptyset$ and $U \cap M_1 = U_1$.

Claim: For any coherent sheaf $\mathcal{G}$ on M, $R^1p_*(p^*\mathcal{G} \otimes \boldsymbol{\mathcal{U}}(i)) = 0$ for $i = 0, 1$.

Note that $R^1 p_* \boldsymbol{\mathcal{U}}(i) = 0$ since $\operatorname{reg}(\mathcal{F}) = 1$ for all $\mathcal{F} \in M$. If $\mathcal{G}$ is locally free the claim follows from the projection formula: $R^1 p_*(p^*\mathcal{G} \otimes \boldsymbol{\mathcal{U}}(i)) = \mathcal{G} \otimes R^1 p_* \boldsymbol{\mathcal{U}}(i)$. Otherwise take a free resolution $\mathcal{E}_\bullet \to \mathcal{G} \to 0$ and split it into short exact sequences. Then apply p^* to the building blocks and tensor them with $\boldsymbol{\mathcal{U}}(i)$. Due to the flatness, exactness is preserved. A look at the long exact sequence associated to p_* finishes the proof of the claim.

In particular, we obtain

$$H^1(U \times \mathbb{P}_3, p^*\mathcal{I} \otimes \boldsymbol{\mathcal{U}}(i)) \cong H^0(U, R^1 p_*(p^*\mathcal{I} \otimes \boldsymbol{\mathcal{U}}(i)) = 0$$

from Leray's spectral sequence and the claim above. It follows that $\widetilde{\varphi}_1$ can be extended:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_1 \arrow[r] & \widetilde{\mathcal{O}}_{U_1} \oplus \widetilde{\mathcal{O}}_{U_1}(-1) \arrow[r, "\widetilde{\varphi}_1"] & \boldsymbol{\mathcal{U}}|_{U_1 \times \mathbb{P}_3} \arrow[r] & 0 \\
& & \widetilde{\mathcal{O}}_U \oplus \widetilde{\mathcal{O}}_U(-1) \arrow[r, "\widetilde{\varphi}"] \arrow[u, two heads, "\mathrm{res}"] & \boldsymbol{\mathcal{U}}|_{U \times \mathbb{P}_3} \arrow[r, dotted] \arrow[u, two heads] & 0.
\end{tikzcd}$$

After possibly shrinking U, we can assume that $\widetilde{\varphi}$ is surjective. Let $\mathcal{K}$ denote the kernel of $\widetilde{\varphi}$. By flatness, $\mathcal{K} \otimes \mathcal{O}_{M_1} = \mathcal{K}_1 = \operatorname{Im}(\mathrm{res}|_{\mathcal{K}})$. The process of extending $\widetilde{\varphi}_1$ can be repeated for the situation

$$\widetilde{\mathcal{O}}_{U_1}(-1) \oplus 3\widetilde{\mathcal{O}}_{U_1}(-2) \xrightarrow{\widetilde{A}_1} \mathcal{K}_1 \to 0$$

in order to obtain the first part of the resolution in the lemma. The last part of the resolution is obtained by the same procedure applied to the kernel of $\widetilde{\mathcal{O}}_{U_1}(-1) \oplus 3\widetilde{\mathcal{O}}_{U_1}(-2) \to \mathcal{K}_1$. $\square$

5.2**. The Parameter space.**
The space of parameters for M will be the space of resolutions described in proposition 3.5. To be precise, let $\mathbb{P} = \mathbb{P}V = \mathbb{P}_3$ and let

$$W \subset \operatorname{Hom}(2\mathcal{O}_{\mathbb{P}}(-3), \mathcal{O}_{\mathbb{P}}(-1) \oplus 3\mathcal{O}_{\mathbb{P}}(-2)) \times \operatorname{Hom}(\mathcal{O}_{\mathbb{P}}(-1) \oplus 3\mathcal{O}_{\mathbb{P}}(-2), \mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(-1))$$

be the locally closed subvariety of pairs (B, A) for which the induced sequence

$$0 \to k^2 \xrightarrow{\widetilde{B}} S^2V^* \oplus (k^3 \otimes V^*) \xrightarrow{\widetilde{A}} S^3V^* \oplus S^2V^*, \quad \widetilde{B} = H^0B(3),\ \widetilde{A} = H^0A(3) \tag{E}$$

is exact. W is acted on by the automorphism group

$$G = \operatorname{Aut}(2\mathcal{O}_{\mathbb{P}}(-3)) \times \operatorname{Aut}(\mathcal{O}_{\mathbb{P}}(-1) \oplus 3\mathcal{O}_{\mathbb{P}}(-2)) \times \operatorname{Aut}(\mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(-1)),$$

the non–reductive group of triples of matrices

$$g_1, \quad g_2 = \left(\begin{array}{c|ccc} \alpha & 0 & 0 & 0 \\ \hline u_1 & & & \\ u_2 & & g & \\ u_3 & & & \end{array}\right), \quad g_3 = \begin{pmatrix} \beta & 0 \\ u & \gamma \end{pmatrix}$$

with $g_1 \in \mathrm{GL}_2(k)$, $g \in \mathrm{GL}_3(k)$, $u, u_i \in V^*$ and $\alpha, \beta, \gamma \in k^*$, the action being given by $(g_1 B g_2^{-1}, g_2 A g_3^{-1})$. The pair $(B, A) \in W$ is called **stable** if

$$A = \begin{pmatrix} z & \lambda \\ q_1 & z_1 \\ q_2 & z_2 \\ q_3 & z_3 \end{pmatrix}$$

satisfies $\lambda \neq 0$ or $z_1 \wedge z_2 \wedge z_3 \neq 0$. We let $X \subset W$ be the open subset of stable pairs.

5.3. **Lemma.** *For any $(B, A) \in X$ the sheaf $\mathcal{F}_A = Coker(A)$ is stable and has the resolution by (B, A) as in proposition 3.5.*

Proof. The pair (B, A) defines the complex

$$0 \to 2\mathcal{O}_{\mathbb{P}}(-3) \xrightarrow{B} \mathcal{O}_{\mathbb{P}}(-1) \oplus 3\mathcal{O}_{\mathbb{P}}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{F}_A \to 0 \qquad (S)$$

which is already exact except possibly at $\mathcal{O}_{\mathbb{P}}(-1) \oplus 3\mathcal{O}_{\mathbb{P}}(-2)$, because $\widetilde{B}$ has rank 2. In order to prove exactness and stability, we distinguish the cases $\lambda \neq 0$ and $\lambda = 0$ for the degree 0 entry of A. If $\lambda \neq 0$, it follows that (B, A) is G–equivalent to a pair

$$\left(\begin{array}{c|ccc} 0 & l_1 & l_2 & l_3 \\ 0 & l_4 & l_5 & l_6 \end{array} \right), \quad \begin{pmatrix} 0 & 1 \\ q_1 & 0 \\ q_2 & 0 \\ q_3 & 0 \end{pmatrix}.$$

In that case the matrix B of rank 2 cannot be equivalent to one with $l_3 = l_6 = 0$, because then $q_1 = q_2 = 0$ and $\widetilde{A}$ would have a higher dimensional kernel. Then the linear part B' of B is a stable point of $\mathrm{Hom}(k^2, k^3 \otimes V^*)$ for the action of $\mathrm{GL}_2(k) \otimes \mathrm{GL}_3(k)$, see [2] and 8.2. It follows that the forms q_i are the minors of this matrix B', that $\mathcal{F}_A \cong \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}}/(q_1, q_2, q_3)\mathcal{O}_{\mathbb{P}}$ is stable and that (S) is exact. If $\lambda = 0$, one proves by an elementary procedure that (B, A) is G–equivalent to a pair

$$\begin{pmatrix} -q_1 & -l_1 & w & 0 \\ -q_2 & -l_2 & 0 & w \end{pmatrix}, \quad \begin{pmatrix} w & 0 \\ 0 & w \\ q_1 & l_1 \\ q_2 & l_2 \end{pmatrix}$$

with $w \wedge l_1 \wedge l_2 \neq 0$. Then also $q_1 l_2 - q_2 l_1 \neq 0$, because otherwise the sequence (E) would not be exact. In this case the sheaf $\mathcal{F}_A$ has support contained in the plane $H = Z(w)$ and has the resolution

$$0 \to 2\mathcal{O}_H(-2) \xrightarrow[\begin{pmatrix} q_1 & l_1 \\ q_2 & l_2 \end{pmatrix}]{} \mathcal{O}_H \oplus \mathcal{O}_H(-1) \to \mathcal{F}_A \to 0.$$

Therefore $\mathcal{F}_A$ is one of the stable sheaves of the moduli space $M_{3m+1}(H)$, see 6.1 and 6.3. Moreover, due to the shape of the matrix pair, the complex (S) is also exact in this case. $\square$

The lemma above shows that $X \longrightarrow M,\ (B,A) \mapsto [\mathcal{F}_A]$ is G–equivariant and surjective. It follows from the existence of the universal family constructed next that the map is also a morphism.

5.4. **The universal sheaf on X.**
On $H_1 := \mathrm{Hom}(2\mathcal{O}_{\mathbb{P}}(-3), \mathcal{O}_{\mathbb{P}}(-1) \oplus 3\mathcal{O}_{\mathbb{P}}(-2))$, we are given the tautological homomorphism

$$k^2 \otimes \mathcal{O}_{H_1} \xrightarrow{b} (S^2V^* \oplus k^3 \otimes V^*) \otimes \mathcal{O}_{H_1}$$

and on $H_2 := \mathrm{Hom}(\mathcal{O}_{\mathbb{P}}(-1) \oplus 3\mathcal{O}_{\mathbb{P}}(-2), \mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(-1))$ we have the tautological maps

$$\begin{array}{lcll} \mathcal{O}_{H_2} & \xrightarrow{a_1} & (V^* \oplus k) \otimes \mathcal{O}_{H_2} & \text{(first component)} \\ k^3 \otimes \mathcal{O}_{H_2} & \xrightarrow{a_2} & (S^2V^* \oplus V^*) \otimes \mathcal{O}_{H_2} & \text{(second component)}\ . \end{array}$$

Combining these three maps with the corresponding evaluation homomorphisms $S^mV^* \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(m)$, we obtain homomorphisms

$$k^2 \otimes \mathcal{O}_{H_1} \boxtimes \mathcal{O}_{\mathbb{P}}(-3) \xrightarrow{\widetilde{B}} \mathcal{O}_{H_1} \boxtimes (\mathcal{O}_{\mathbb{P}}(-1) \oplus k^3 \otimes \mathcal{O}_{\mathbb{P}}(-2))$$

and

$$\mathcal{O}_{H_2} \boxtimes (\mathcal{O}_{\mathbb{P}}(-1) \oplus k^3 \otimes \mathcal{O}_{\mathbb{P}}(-2)) \xrightarrow{\widetilde{A}} \mathcal{O}_{H_2} \boxtimes (\mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(-1)).$$

Finally, restricting to $X \times \mathbb{P} \subset W \times \mathbb{P} \subset H_1 \times H_2 \times \mathbb{P}$ and using the notation $\widetilde{\mathcal{O}}_X(m) = \mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}}(m)$, we get the complex

$$0 \to 2\widetilde{\mathcal{O}}_X(-3) \xrightarrow{\widetilde{B}} \widetilde{\mathcal{O}}_X(-1) \oplus 3\widetilde{\mathcal{O}}_X(-2) \xrightarrow{\widetilde{A}} \widetilde{\mathcal{O}}_X \oplus \widetilde{\mathcal{O}}_X(-1) \to \boldsymbol{\mathcal{F}} \to 0 \qquad (R)$$

with $\boldsymbol{\mathcal{F}} = \mathrm{Coker}(\tilde{A})$. It is exact after restriction to a point (B,A) of X, giving $\boldsymbol{\mathcal{F}}_{(B,A)} = \mathrm{Coker}(A) = \mathcal{F}_A$. This proves that the complex (R) is exact and that $\boldsymbol{\mathcal{F}}$ is a flat family of sheaves over X. Now the above family defines a unique morphism $X \overset{\mu}{\twoheadrightarrow} M$ such that $\boldsymbol{\mathcal{F}} \cong \mu_{\mathbb{P}}^* \boldsymbol{\mathcal{U}}$, whose underlying map is $(B,A) \mapsto [\mathcal{F}_A]$, which is surjective. It is easy to verify that the fibers of μ coincide with the G–orbits.

Remark: The variety X consists of two irreducible components $X_1 = \mu^{-1}(M_1)$ and $X_0 = \mu^{-1}(M_0)$ with $X_1 = \{(B,A) \in X \mid \lambda = 0\}$ and X_0 the closure of $X \setminus X_1$.

5.5. **Theorem.** *The morphism $X \xrightarrow{\mu} M$ is a geometric quotient of X by the (non–reductive) group G in the sense of Geometric Invariant Theory, see* [9], [1].

Proof. Since the fibers of μ coincide with the G–orbits, it is sufficient to prove that μ admits local sections (slices) through any point (B,A) in X. Let $(B,A) \in X$ be given and let $a := [\mathcal{F}_A] \in M$. By lemma 5.1 there is an open neighbourhood $U(a) \subset M$ with a resolution

$$0 \to 2\widetilde{\mathcal{O}}_U(-3) \xrightarrow{\Psi} \widetilde{\mathcal{O}}_U(-1) \oplus 3\widetilde{\mathcal{O}}_U(-2) \xrightarrow{\Phi} \widetilde{\mathcal{O}}_U \oplus \widetilde{\mathcal{O}}_U(-1) \to \boldsymbol{\mathcal{U}}|_{U \times \mathbb{P}} \to 0$$

of the universal sheaf. Define a morphism $s : U \to X$ by $b \mapsto (\Psi(b), \Phi(b))$. Since $\boldsymbol{\mathcal{U}}_a \cong \mathcal{F}_A$ we have $\Phi(a) \sim A$. Then there is an element $g \in G$ with $g\,.\,s(a) = (B,A)$ and $g\,.\,s$ is then a section of μ through (B,A). $\square$

Remark 1: The definition of a geometric quotient in [9] does not include that the quotient map has to be affine. In our case we do not know whether the action is proper or whether the quotient map is affine.

Remark 2: There is no common stabilizer for all the points of X. The stabilizers for the points of X_1 are all the same and isomorphic to $k^* \times k$, whereas the stabilizers for the points of $X_0 \smallsetminus X_1$ are all isomorphic to $k^* \times k^*$. Therefore, the action of G modulo a subgroup cannot be free and $X \xrightarrow{\mu} M$ is not a principal fibration.

Remark 3: In [1] polarizations Λ have been introduced by which open sets $W^s(G, \Lambda) \subset W^{ss}(G, \Lambda) \subset W$ of stable and semi-stable points for a *non–reductive* group G can be defined in such a way that $W^{ss}(G, \Lambda)$ admits a good and projective quotient $W^{ss}(G, \Lambda)//G$. Moreover, $W^s(G, \Lambda)$ and closed invariant subsets of $W^s(G, \Lambda)$ have a geometric quotient.

In the above case $X \subset W$, one can prove that there exists no polarization Λ in the sense of [1], such that X is a closed subset of $W^s(G, \Lambda)$, cf. [3].

6. The component M_1

Intuitively, the component M_1 of $M_{3m+1}(\mathbb{P}_3)$ should be fibered by the schemes $M_{3m+1}(H)$ of stable sheaves on a plane H in $\mathbb{P}_3$, cf. proof of lemma 5.3. In fact, M_1 is isomorphic to the relative Simpson scheme over $\mathbb{P}_3^*$ as we will see. Therefore, we first digress to the moduli space $M_{3m+1}(\mathbb{P}_2)$, see also [8].

6.1. The moduli space $M_{3m+1}(\mathbb{P}_2)$.

Let $M(\mathbb{P}_2) = M_{3m+1}(\mathbb{P}_2)$. Because of the multiplicity 3, this scheme is a fine moduli space with a universal sheaf $\boldsymbol{\mathcal{V}}$ over $M(\mathbb{P}_2) \times \mathbb{P}_2$ and it consists entirely of stable sheaves. By lemma 3.3, any sheaf $\mathcal{F}$ in $M(\mathbb{P}_2)$ has a resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow[\begin{pmatrix} q_1 & l_1 \\ q_2 & l_2 \end{pmatrix}]{} \mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \to \mathcal{F} \to 0 \qquad (PR)$$

with independent linear forms l_1, l_2 and quadratic forms q_1, q_2 such that $f = q_1 l_2 - q_2 l_1 \neq 0$. The plane cubic $C = Z(\mathcal{F})$ has the equation f.

6.2. Lemma. *With the above notation let $p \in C$ be the point determined by $l_1(p) = l_2(p) = 0$. Then the following are equivalent for the stalk $\mathcal{F}_p$.*

(i) *$\mathcal{F}_p$ is not free.*
(ii) *p is a singular point of C.*
(iii) *q_1 and q_2 vanish both at p.*

The elementary proof is left to the reader.

Remark: This lemma shows that the sheaf $\mathcal{F}$ is a line bundle on C, even if the support C is singular, if and only if the point p is not a singular point. If p is a singular point, $\mathcal{F}$

is a Cohen–Macaulay sheaf on C of rank 1. We call the sheaves $\mathcal{F}$ in $M(\mathbb{P}_2)$ that are not locally free on their support **singular** sheaves.

Now we consider a parameter space Y for $M(\mathbb{P}_2)$, using the same method as in section 5. We define Y to be the quasi–affine variety of 2×2 matrices as in (PR). It is acted on by the group

$$G = \mathrm{Aut}(2\mathcal{O}_{\mathbb{P}V}(-2)) \times \mathrm{Aut}(\mathcal{O}_{\mathbb{P}V} \oplus \mathcal{O}_{\mathbb{P}V}(-1)) = \mathrm{GL}_2(k) \times \left\{ \begin{pmatrix} \alpha & z \\ 0 & \beta \end{pmatrix} \;\middle|\; \alpha\beta \neq 0, z \in V^* \right\}$$

via $(g,h)\,.\,A = gAh^{-1}$.

It is clear from the description (PR) of the sheaves $\mathcal{F}$ in $M(\mathbb{P}_2)$ that the isomorphism classes $[\mathcal{F}]$ correspond bijectively to the orbits of this action on Y. So we expect that $Y/G = M(\mathbb{P}_2)$, if the quotient exists. Indeed, as in 5.4 we can construct a universal sheaf $\boldsymbol{\mathcal{G}}$ over $Y \times \mathbb{P}V$ with a presentation

$$0 \to k^2 \otimes \mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}V}(-2) \xrightarrow{\Phi} \mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}V} \oplus \mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}V}(-1) \to \boldsymbol{\mathcal{G}} \to 0.$$

For any $y \in Y$, the restricted homomorphism $\Phi_y = A$ is one of the matrices above and $\boldsymbol{\mathcal{G}}_y$ belongs to $M(\mathbb{P}_2)$. Thus $\boldsymbol{\mathcal{G}}$ is a flat family and defines a surjective morphism $Y \xrightarrow{\nu} M(\mathbb{P}_2)$ with $\boldsymbol{\mathcal{G}} \cong \nu^*_{\mathbb{P}_2}\boldsymbol{\mathcal{V}}$. By the same method as in the proof of theorem 5.5 one can directly construct local slices of ν using the relative Beilinson resolution of the universal sheaf $\boldsymbol{\mathcal{V}}$, which is locally free in this case. We so obtain the

6.3. **Proposition.** *The scheme $M(\mathbb{P}_2) = M_{3m+1}(\mathbb{P}_2)$ together with the morphism $Y \xrightarrow{\nu} M(\mathbb{P}_2)$ is a geometric quotient of Y by the action of the (non-reductive) group G.*

Let now $Z \subset \mathbb{P}S^3V^* \times \mathbb{P}V$ denote the universal cubic given by pairs $(\langle f\rangle, \langle x\rangle)$ with $f(x) = 0$. The map

$$\begin{pmatrix} q_1 & l_1 \\ q_2 & l_2 \end{pmatrix} \quad\mapsto\quad (\langle q_1 l_2 - q_2 l_1\rangle, \langle l_1 \wedge l_2\rangle)$$

is a surjective morphism $Y \xrightarrow{\gamma} Z$, where we identify $\langle l_1 \wedge l_2\rangle$ with the point p defined by $l_1(p) = l_1(p) = 0$ via $V \cong \Lambda^2 V^*$. It is not difficult to see that γ is a geometric quotient, cf. also [1]. Together with proposition 6.3, this implies

6.4. **Corollary.** *$M_{3m+1}(\mathbb{P}_2)$ is isomorphic to the universal cubic $Z \subset \mathbb{P}S^3V^* \times \mathbb{P}V$.*

Remark 1: The isomorphism $M_{3m+1}(\mathbb{P}_2) \cong Z$ has already been mentioned in [8].

Remark 2: A universal sheaf $\boldsymbol{\mathcal{F}}$ on $Z \times \mathbb{P}_2$ can also directly be defined as follows. Let $H = \mathbb{P}S^3V^*$. The dual of the ideal sheaf sequence $0 \to \mathcal{I}_\Delta \to \mathcal{O}_{Z\times_H Z} \to \mathcal{O}_\Delta \to 0$ of the diagonal $\Delta \subset Z \times_H Z$ is

$$0 \to \mathcal{O}_{Z\times_H Z} \to \boldsymbol{\mathcal{F}} \to \mathcal{E}xt^1(\mathcal{O}_\Delta, \mathcal{O}_{Z\times_H Z}) \to 0.$$

The $\mathcal{E}xt$–sheaf is in this case isomorphic to $\mathcal{O}_\Delta$ and the sequence does not split. For a single cubic we get back the sequence $0 \to \mathcal{O}_C \to \mathcal{F}_C \to k_p \to 0$, see lemma 3.2. This construction is motivated by the classical construction of the Poincaré bundle for a single elliptic curve. Using the embedding $Z \times_H Z \subset Z \times \mathbb{P}_2$ determined by $((C,x),(C,y)) \mapsto ((C,x),y)$, the sheaf $\boldsymbol{\mathcal{F}}$ can be considered as a sheaf on $Z \times \mathbb{P}_2$ which is flat over Z. It is isomorphic to the universal sheaf.

Remark 3: One has $Y = W^s(G,\Lambda) = W^{ss}(G,\Lambda)$ for the polarization $\Lambda = (\frac{1}{2}; \frac{3}{4}, \frac{1}{4})$ in the sense of [1], cf. [3].

6.5**. The relative case.**
We prove now that the component M_1 is fibered by the spaces $M_{3m+1}(H)$, H a plane in $\mathbb{P}_3$. Let $S := \mathbb{P}_3^* = \mathbb{P}V^*$ and let $\mathbb{P}\mathcal{H} \to S$ be the bundle of planes, induced by the tautological subbundle $\mathcal{H} \subset V \otimes \mathcal{O}_S$. We denote by

$$M_S = M_{3m+1}(\mathbb{P}\mathcal{H}/S) \to S$$

the relative Simpson scheme, such that each fiber $M_s \cong M_{3m+1}(\mathbb{P}\mathcal{H}_s)$. See [6], theorem 4.3.7, for the general situation. This scheme M_S is again a fine moduli space with a universal sheaf

$$\boldsymbol{\mathcal{V}}_S \quad \text{over} \quad M_S \times_S \mathbb{P}\mathcal{H} \subset M_S \times \mathbb{P}V.$$

Since $\mathbb{P}\mathcal{H}$ is locally trivial over S, also M_S is locally trivial over S with fiber $M_{3m+1}(\mathbb{P}_2)$. Corollary 6.4 implies that M_S is smooth, irreducible and rational of dimension 13. Consider now the relative universal cubic

$$Z_S \subset \mathbb{P}(S^3\mathcal{H}^*) \times_S \mathbb{P}\mathcal{H}.$$

It is easy to show that there exists an isomorphism $M_S \cong Z_S$ which extends the isomorphisms $M_s \cong Z_s$ of the fibers from 6.4.

6.6**. Proposition.** *The subscheme $M_1 \subset M$ is isomorphic to M_S and Z_S and therefore smooth, rational of dimension* 13.

Proof. Let $\boldsymbol{\mathcal{U}}_1$ be the restriction of the universal sheaf $\boldsymbol{\mathcal{U}}$ to $M_1 \times \mathbb{P}V$. It has a presentation

$$\mathcal{A}_1 \boxtimes \mathcal{O}_{\mathbb{P}V}(-1) \oplus \mathcal{B}_1 \boxtimes \mathcal{O}_{\mathbb{P}V}(-2) \xrightarrow{A} \mathcal{A}_0 \boxtimes \mathcal{O}_{\mathbb{P}V} \oplus \mathcal{B}_0 \boxtimes \mathcal{O}_{\mathbb{P}V}(-1) \to \boldsymbol{\mathcal{U}}_1 \to 0$$

with locally free sheaves $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_0, \mathcal{B}_0$ on M_1 of ranks $1, 3, 1, 1$ respectively, see proposition 3.6. Note that the component $\mathcal{A}_1 \boxtimes \mathcal{O}_{\mathbb{P}V}(-1) \to \mathcal{B}_0 \boxtimes \mathcal{O}_{\mathbb{P}V}(-1)$ of the homomorphism A vanishes identically and that the component $\mathcal{A}_1 \boxtimes \mathcal{O}_{\mathbb{P}V}(-1) \to \mathcal{A}_0 \boxtimes \mathcal{O}_{\mathbb{P}V}$ corresponds to a map

$$\mathcal{O}_{M_1} \xrightarrow{\omega} \mathcal{A}_0 \otimes \mathcal{A}_1^* \otimes V^*.$$

This section ω of $\mathcal{A}_0 \otimes \mathcal{A}_1^* \otimes V^*$ is nowhere zero, because A can be expressed locally by a matrix

$$\begin{pmatrix} w & 0 \\ q_1 & z_1 \\ q_2 & z_2 \\ q_3 & z_3 \end{pmatrix}$$

of linear and quadratic forms where w is the local expression of ω. But the linear form w is nowhere vanishing because the cokernel sheaves are supported in M_1. So ω induces a morphism

$$M_1 \xrightarrow{\omega} \mathbb{P}V^* = S,$$

which is locally given by $m \mapsto \langle w(m) \rangle$, where $w(m)$ is the equation of the plane which contains the support of $(\boldsymbol{\mathcal{U}}_1)_m$. It is now clear from this description that $\boldsymbol{\mathcal{U}}_1$ is also a relative sheaf. Therefore there exists a unique morphism

$$\begin{tikzcd} M_1 \arrow[rr, "f"] \arrow[dr] & & M_S \arrow[dl] \\ & S & \end{tikzcd}$$

with $\boldsymbol{\mathcal{U}}_1 \cong f_{\mathbb{P}V}^* \boldsymbol{\mathcal{V}}_S$. Conversely, $\boldsymbol{\mathcal{V}}_S$ is a flat family of stable sheaves with Hilbert polynomial $3m+1$ on $M_S \times \mathbb{P}V$ and thus defines a morphism $M_S \xrightarrow{g} M$ with $\boldsymbol{\mathcal{V}}_S \cong g_{\mathbb{P}V}^* \boldsymbol{\mathcal{U}}$. Now g factors through M_1 and hence $\boldsymbol{\mathcal{V}}_S \cong g_{\mathbb{P}V}^* \boldsymbol{\mathcal{U}}_1$. It follows that f and g are inverse to each other. $\square$

7. Deformations and tangent spaces

In this section we show that the elements of the intersection of the two components M_0 and M_1 are exactly the singular planar sheaves in M_1, i.e. the planar sheaves which are not locally free on their support. Furthermore, the two components, defined as reduced subschemes, meet transversally.

Let $M_1' \subset M_1$ denote the locus of singular sheaves. By lemma 6.2 and proposition 3.5 any $\mathcal{F}$ in M_1' has a resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_3}(-3) \xrightarrow{B_0} \mathcal{O}_{\mathbb{P}_3}(-1) \oplus 3\mathcal{O}_{\mathbb{P}_3}(-2) \xrightarrow{A_0} \mathcal{O}_{\mathbb{P}_3} \oplus \mathcal{O}_{\mathbb{P}_3}(-1) \to \mathcal{F} \to 0$$

with

$$B_0 = \begin{pmatrix} -q_1 & -l_1 & w & 0 \\ -q_2 & -l_2 & 0 & w \end{pmatrix} \quad \text{and} \quad A_0 = \begin{pmatrix} w & 0 \\ 0 & w \\ q_1 & l_1 \\ q_2 & l_2 \end{pmatrix}$$

and $q_i = a_i\, l_1 + b_i\, l_2$ for suitable linear forms a_i, b_i, $i = 1, 2$.

Using the same method as in the proof of 5.1 one can prove that for any flat deformation $\boldsymbol{\mathcal{F}} \in \mathrm{Coh}(V \times \mathbb{P}_3)$ of $\mathcal{F} = \boldsymbol{\mathcal{F}}_0$ over an open neighbourhood $V \subset \mathbb{A}^1$ of 0 there is an open neighbourhood $U \subset V$ over which $\boldsymbol{\mathcal{F}}$ can be represented by a resolution

$$0 \to 2\widetilde{\mathcal{O}}_U(-3) \xrightarrow{B_0+tB_1} \widetilde{\mathcal{O}}_U(-1) \oplus 3\widetilde{\mathcal{O}}_U(-2) \xrightarrow{A_0+tA_1} \widetilde{\mathcal{O}}_U \oplus \widetilde{\mathcal{O}}_U(-1) \to \boldsymbol{\mathcal{F}} \to 0,$$

where $\widetilde{\mathcal{O}}_U(d) := \mathcal{O}_U \boxtimes \mathcal{O}_{\mathbb{P}_3}(d)$ and where B_1 and A_1 are matrices which may depend on the parameter t of $\mathbb{A}^1$. (To see this, note that $H^1(\mathbb{P}_3, \mathcal{F}) = H^1(\mathbb{P}_3, \mathcal{F}(1)) = 0$ and that then also the direct images $R^1p_*\boldsymbol{\mathcal{F}}, R^1p_*\boldsymbol{\mathcal{F}}(1)$ vanish on a neighbourhood U of 0 by the base change theorem. Assuming U to be affine after shrinking, also $H^1(U \times \mathbb{P}_3, \boldsymbol{\mathcal{F}}) = H^1(U \times \mathbb{P}_3, \boldsymbol{\mathcal{F}}(1)) = 0$. This implies that the homomorphism A_0 can be lifted and that its lifting is still surjective after eventually shrinking U. Then use the same argument for the kernels.)

7.1. **Lemma.** *$M_1' \subset M_0 \cap M_1$.*

Proof. Let $[\mathcal{F}] \in M_1'$ be arbitrary with corresponding matrices A_0 and B_0 as above. The ideal of the support $Z(\mathcal{F})$ is given by $(q_1l_2 - q_2l_1, w)$.

We construct explicitly a deformation $\boldsymbol{\mathcal{F}}$ of the sheaf $\mathcal{F}$ over $\mathbb{A}^1$ with $\boldsymbol{\mathcal{F}}_0 \cong \mathcal{F}$ and $\boldsymbol{\mathcal{F}}_t \in M_0 \smallsetminus M_1$ for $t \neq 0$ by choosing the following first order deformation matrices:

$$B_1 = \begin{pmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_2 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} a_1 + b_2 & 1 \\ b_1a_2 - b_2a_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Consider the diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & 2\,\widetilde{\mathcal{O}}(-3) \arrow[r, "B_t=B_0+t\,B_1"] \arrow[d, "T_2", "\approx"'] \arrow[dr, phantom, "\circlearrowleft"] & \widetilde{\mathcal{O}}(-1) \oplus 3\,\widetilde{\mathcal{O}}(-2) \arrow[r, "A_t=A_0+t\,A_1"] \arrow[d, "\approx", "T_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \widetilde{\mathcal{O}} \oplus \widetilde{\mathcal{O}}(-1) \arrow[r] \arrow[d, "\approx", "T_0"'] \arrow[dr, phantom, "\circlearrowleft"] & \boldsymbol{\mathcal{F}} \arrow[r] \arrow[d, dotted, "\approx"] & 0 \\
t\neq 0 & 0 \arrow[r, dotted] & 2\,\widetilde{\mathcal{O}}(-3) \arrow[r, dotted, "\widetilde{B}_t"'] & \widetilde{\mathcal{O}}(-1) \oplus 3\,\widetilde{\mathcal{O}}(-2) \arrow[r, dotted, "\widetilde{A}_t"'] & \widetilde{\mathcal{O}} \oplus \widetilde{\mathcal{O}}(-1) \arrow[r, dotted] & \boldsymbol{\mathcal{G}} \arrow[r, dotted] & 0
\end{tikzcd}$$

It is straightforward to check that the first row in the diagram is exact *for all* $t \in \mathbb{A}_1$. Thus, the cokernel of A_t is our $\mathbb{A}_1$-flat deformation $\boldsymbol{\mathcal{F}}$. For $t \neq 0$, we choose the transformation matrices T_i in the diagram as:

$$T_{0,t} = \begin{pmatrix} 1 & 0 \\ -a_1 - b_2 - t^{-1}w & -t^{-2} \end{pmatrix}, \quad T_{1,t} = \begin{pmatrix} -t & 0 & 0 & 0 \\ -w & t & 0 & 0 \\ -l_1 & 0 & t & 0 \\ -l_2 & 0 & 0 & t \end{pmatrix} \quad \text{and} \quad T_{2,t} = \begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}$$

Using those, we obtain the matrices in the second row:

$$\widetilde{B}_t = \begin{pmatrix} 0 & -l_1 & ta_1 + w & tb_1 \\ 0 & -l_2 & ta_2 & tb_2 + w \end{pmatrix}, \quad \widetilde{A}_t = \begin{pmatrix} 0 & 1 \\ t^2b_1a_2 - t^2b_2a_1 - ta_1w - tb_2w - w^2 & 0 \\ tq_1 - ta_1l_1 - tb_2l_1 - l_1w & 0 \\ tq_2 - ta_1l_2 - tb_2l_2 - l_2w & 0 \end{pmatrix}$$

Again, one can verify the exactness of that row for $t \neq 0$. Consequently, we get as cokernel of $\widetilde{A}_t$ a family $\boldsymbol{\mathcal{G}} \in \mathrm{Coh}(\mathbb{P}_3 \times \mathbb{A}_1)$ with $\boldsymbol{\mathcal{F}}_t \cong \boldsymbol{\mathcal{G}}_t = \mathcal{O}_{C_t}$, $t \neq 0$ where the curves C_t are given by the ideal $(\,t^2b_1a_2 - t^2b_2a_1 - ta_1w - tb_2w - w^2,\ tq_1 - ta_1l_1 - tb_2l_1 - l_1w,\ tq_2 - ta_1l_2 - tb_2l_2 - l_2w\,)$. Therefore the isomorphism classes $[\boldsymbol{\mathcal{F}}_t]$ are elements of $M \smallsetminus M_1 \cong H_0 \smallsetminus H_1$ for non-vanishing t. □

In order to show that in fact $M_1' = M_0 \cap M_1$, we determine the dimension of the tangent spaces at different points of M. This will also be used for the proof of the transversality.

7.2. **Proposition.** *Let $\mathcal{F}$ be any sheaf in M.*

(i) *If $\mathcal{F} \in M \smallsetminus M_1$, then* $\dim T_{\mathcal{F}}M = 12$.
(ii) *If $\mathcal{F} \in M_1 \smallsetminus M_1'$, then* $\dim T_{\mathcal{F}}M = 13$.
(iii) *If $\mathcal{F} \in M_1'$, then* $\dim T_{\mathcal{F}}M = 14$.

Preparations. The parameter space X defined in section 5 is reduced. Therefore, also M is reduced as a geometric quotient. X consists of matrix pairs (B, A) with $B \circ A = 0$ and $\mathrm{rk}(\widetilde{B}) = 2$.

Let $p := (B_0, A_0) \in X$ and denote by $F_p = G.p$ the G–orbit through p which coincides with the fiber of $\mathcal{F} := \mu(p)$ under the map $\mu : X \to M$. Standard arguments show that

$$\begin{aligned} T_pX &= \{\,(B_1, A_1) \in X \mid B_1A_0 + B_0A_1 = 0\,\}, \\ T_pF_p &= \{\,(RB_0 - B_0S, SA_0 - A_0T) \mid (R, S, T) \in \mathrm{Lie}(G)\,\}. \end{aligned}$$

Every $[\mathcal{G}] \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ has a resolution

$$0 \longrightarrow \mathcal{L}_2 \oplus \mathcal{L}_2 \xrightarrow{\begin{pmatrix} B_0 & 0 \\ B_1 & B_0 \end{pmatrix}} \mathcal{L}_1 \oplus \mathcal{L}_1 \xrightarrow{\begin{pmatrix} A_0 & 0 \\ A_1 & A_0 \end{pmatrix}} \mathcal{L}_0 \oplus \mathcal{L}_0 \longrightarrow \mathcal{G} \longrightarrow 0,$$

obtained by adding two copies of the resolution 3.5 of $\mathcal{F}$. Moreover,

- $\begin{pmatrix} B_0 & 0 \\ B_1 & B_0 \end{pmatrix}\begin{pmatrix} A_0 & 0 \\ A_1 & A_0 \end{pmatrix} = 0$ is equivalent to $B_1A_0 + A_0B_1 = 0$.
- $\mathcal{G} \cong \mathcal{F} \oplus \mathcal{F}$ if and only if $A_1 = SA_0 - A_0T$ and $B_1 = RB_0 - B_0T$ for some matrices S, T, R.

Therefore we get an exact sequence $0 \to T_pF_p \longrightarrow T_pX \longrightarrow \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \to 0$, where the second map is given by $(B_1, A_1) \mapsto [\mathcal{G}]$. Since the Zariski tangent space $T_{\mathcal{F}}M$ is isomorphic to the extension group $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$, cf. proposition 2.2, we obtain

$$T_{\mathcal{F}}M \cong T_pX / T_pF_p. \tag{$*$}$$

Proof. Part (i) is an immediate consequence of lemma 4.1: $M \smallsetminus M_1 \cong H_0 \smallsetminus H_1$ is smooth of dimension 12. For (ii) and (iii) consider $X_1 = \mu^{-1}(M_1) = \{\,(B, A) \in X \mid \lambda(A) = 0\,\}$, where $\lambda(A)$ is the degree 0 entry of A. Let X_1' denote the preimage of the subset $\mu^{-1}(M_1')$ of singular sheaves in M_1. Clearly, $T_pX_1 \subset T_pX$ for $p \in X_1$.

(1) For $p = (B_0, A_0) \in X_1 \smallsetminus X_1'$, we have $T_pX_1 = T_pX$. To see this, let $(B_1, A_1) \in T_pX$. The equation $B_1A_0 + B_0A_1 = 0$ can be written more explicitly as

$$\begin{pmatrix} r_1 & x_1 & x_3 & x_5 \\ r_2 & x_2 & x_4 & x_6 \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & w \\ q_1 & l_1 \\ q_2 & l_2 \end{pmatrix} + \begin{pmatrix} -q_1 & -l_1 & w & 0 \\ -q_2 & -l_2 & 0 & w \end{pmatrix} \begin{pmatrix} u & \lambda \\ h & v \\ h_1 & v_1 \\ h_2 & v_2 \end{pmatrix} = 0.$$

The two entries $(1, 2)$ and $(2, 2)$ of this matrix modulo the three linear forms w, l_1 and l_2 yield the equations

$$\overline{q}_1\lambda = 0 \quad \text{and} \quad \overline{q}_2\lambda = 0.$$

Since $p \notin X_1'$, the quadratic forms q_1 and q_2 are not contained in the ideal (l_1, l_2, w) and therefore $\overline{q}_1$ and $\overline{q}_2$ cannot both be equal to zero. Hence $\lambda = 0$ and $(B_1, A_1) \in T_pX_1$.

(2) On the other hand, if $p \in X_1'$ there exists in addition the tangent vector

$$(B_1, A_1) = \left(\begin{pmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_2 \end{pmatrix}, \begin{pmatrix} a_1 + b_2 & 1 \\ b_1a_2 - b_2a_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

from lemma 7.1 which is *not* contained in T_pX_1. A direct computation shows that T_pX_1 and the vector (B_1, A_1) span T_pX. Therefore $\dim T_pX = \dim T_pX_1 + 1$ in this case. Since $F_p \subset X_1$ for all $p \in X_1$, we obtain from $(*)$ that

$$T_{\mathcal{F}}M/T_{\mathcal{F}}M_1 \cong T_pX/T_pX_1.$$

According to proposition 6.6, M_1 is smooth of dimension 13, and we get

$$\dim T_{\mathcal{F}}M = \dim T_{\mathcal{F}}M_1 + \dim T_pX/T_pX_1 = \begin{cases} 14 & \text{if } \mathcal{F} \in M_1' \\ 13 & \text{if } \mathcal{F} \in M_1 \smallsetminus M_1'. \end{cases}$$

□

7.3. **Corollary.** *The intersection $M_0 \cap M_1$ coincides with the set M_1' of singular planar sheaves.*

Proof. By lemma 7.1, $M_1' \subset M_0 \cap M_1$. Then $\dim T_{\mathcal{F}}M > \dim M = 13$ for any $\mathcal{F}$ in $M_0 \cap M_1 \smallsetminus M_1'$. By proposition 7.2 (ii), this is not possible. □

7.4. **Proposition.** *For any point $p \in M_0 \cap M_1$,*

(a) $T_pM_0 + T_pM_1 = T_pM$
(b) $T_p(M_0 \cap M_1) = T_pM_0 \cap T_pM_1$

Proof. We have $T_pM_0 + T_pM_1 \subset T_pM$ and therefore $13 \le \dim(T_pM_0 + T_pM_1) \le \dim T_pM = 14$ by proposition 7.2. If $\dim(T_pM_0 + T_pM_1) = 13$, then $T_pM_0 \subset T_pM_1$ since $\dim T_pM_1 = 13$ because of the smoothness of M_1. But this is not possible because there are deformation paths from p to points $q \in M_0 \smallsetminus M_1$ as we saw in the proof of lemma 7.1. Part (b) is a general fact for intersections of subvarieties. □

Remark: We will show in section 8 that $M_0 \cap M_1$ is smooth of dimension 11. Then it follows already from (a) and (b) that M_0 is also smooth along $M_0 \cap M_1$ and that the components M_0 and M_1 intersect transversally.

8. The component M_0 and transversality

We will now prove that the isomorphism $M_0 \setminus M_1 \cong H_0 \setminus H_1$ from lemma 4.1 extends to an isomorphism $M_0 \cong H_0$ and that moreover M_0 is isomorphic to a blowup of the smooth variety N of nets of quadrics as described in [2], see 8.2 below. Each of these two statements implies that M_0 is smooth and rational of dimension 12 due to the results in [11] and [2]. The blowup structure of H_0 has been mentioned without proof in [2]. We include a nice explicit description for M_0 using the deformation matrices of section 7.

8.1. **Proposition.** *The component M_0 is isomorphic to the component H_0 of the Hilbert scheme $Hilb_{3m+1}(\mathbb{P}_3)$.*

Proof. We extend the isomorphism $f' : M_0 \setminus M_1 \to H_0 \setminus H_1$ using Fitting ideals. The Fitting ideal of a sheaf $\mathcal{F} \in M$ is the ideal $\mathrm{Fitt}(\mathcal{F})$ generated by the 2–minors of the presentation matrix A in the resolution of $\mathcal{F}$. Recall from proposition 3.5 that A has the form

$$\begin{pmatrix} w & 0 \\ 0 & w \\ q_1 & l_1 \\ q_2 & l_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ q_1 & 0 \\ q_2 & 0 \\ q_3 & 0 \end{pmatrix}$$

depending on whether the support $Z(\mathcal{F})$ is contained in a plane or not. By corollary 7.3, the elements of $M_0 \cap M_1$ are exactly the singular planar sheaves which are characterized by matrices A of the first type with the additional condition $(q_1, q_2) \subset (l_1, l_2)$. Therefore we obtain

$$\begin{aligned} \mathrm{Fitt}(\mathcal{F}) &= (q_1, q_2, q_3) \quad \text{if } \mathcal{F} \in M_0 \setminus M_1 \\ \mathrm{Fitt}(\mathcal{F}) &= (w^2, wl_1, wl_2, l_1q_2 - l_2q_1) \quad \text{if } \mathcal{F} \in M_0 \cap M_1 \end{aligned}$$

Note, that on the other hand $(w^2, wl_1, wl_2, l_1q_2 - l_2q_1)$ is exactly the normal form for ideals of cubics in $H_0 \cap H_1$, see [11]. Therefore we get a surjective, set-theoretical map $f : M_0 \longrightarrow H_0$, $\mathcal{F} \mapsto \mathcal{O}_{Z(\mathrm{Fitt}(\mathcal{F}))}$ extending the isomorphism f'. It is indeed bijective, the injectivity on $M_0 \cap M_1$ is easy to check. Applying $\mathcal{O}_{Z(\mathrm{Fitt}(\,\cdot\,))}$ to each fiber $\mathcal{F} = \boldsymbol{\mathcal{U}}_{\mathcal{F}}$ of the universal sheaf $\boldsymbol{\mathcal{U}}|_{M_0 \times \mathbb{P}_3}$, we obtain a family $\boldsymbol{\mathcal{G}} \in \mathrm{Coh}(H_0 \times \mathbb{P}_3)$. It is straightforward to see that $\boldsymbol{\mathcal{G}}$ is H_0–flat with constant Hilbert polynomial $3m + 1$ along the fibers. The morphism induced by $\boldsymbol{\mathcal{G}} \in \mathcal{M}_{3m+1}(\mathbb{P}_3)(H_0)$ is exactly the set-theoretical map f. Since H_0 is smooth the bijective morphism f is an isomorphism by Zariski's main theorem. $\square$

Remark: The above Fitting map cannot be extended to a morphism $M_{3m+1}(\mathbb{P}_3) \to \mathrm{Hilb}_{3m+1}(\mathbb{P}_3)$, because the family of Fitting ideals of all the matrices A is not flat over M.

8.2. **Nets of quadrics.**
Let $V \cong k^4$ and let N denote the geometric quotient

$$\mathrm{Hom}(k^2, k^3 \otimes V^*)^s / \mathrm{GL}_2(k) \times GL_3(k)$$

as described in [2]. An adhoc definition of the stability of a matrix

$$Q = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

in $\mathrm{Hom}(k^2, k^3 \otimes V^*)$ is that Q is not equivalent to a matrix with two zeros in a row or in a column. The scheme N is projective, smooth, rational of dimension 12. Its tangent space at $[Q]$ is isomorphic to

$$T_{[Q]}N \cong \mathrm{Hom}(k^2, k^3 \otimes V^*)/F_Q \qquad \text{with} \quad F_Q = \{SQ - QR \mid \quad S \in \mathfrak{gl}_2, R \in \mathfrak{gl}_3\},$$

as in the proof of proposition 7.2. Let $N_1 \subset N$ be the subvariety whose points are of the form

$$\begin{bmatrix} l_1 & w & 0 \\ l_2 & 0 & w \end{bmatrix}$$

with linear independent forms $w, l_1, l_2 \in V^*$. N_1 is the subvariety of classes of matrices, whose 3 minors have a common linear factor and thus do not determine a cubic space curve. On the other hand

$$N \smallsetminus N_1 \cong H_0 \smallsetminus H_1 \cong M \smallsetminus M_1$$

because the 3 quadrics of a matrix in $N \smallsetminus N_1$ generate the ideal of a cubic space curve in $\mathbb{P}_3 = \mathbb{P}V$, see [2].

Let $q \in N_1$ be defined by the matrix $Q = \left(\begin{smallmatrix} l_1 & w & 0 \\ l_2 & 0 & w \end{smallmatrix}\right)$ and let l_3 be a fourth linear form independent of w, l_1, l_2, and let

$$T'_Q \subset \mathrm{Hom}(k^2, k^3 \otimes V^*)$$

be the 5–dimensional subspace spanned by the matrices

$$\left(\begin{array}{c|c|c} \rho l_3 & \lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 & 0 \\ \hline \sigma l_3 & 0 & \lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 \end{array}\right)$$

with $\rho, \sigma, \lambda_1, \lambda_2, \lambda_3 \in k$. We have the

8.3. **Lemma.** (i) $T'_Q \cap F_Q = 0$
(ii) $T_q N_1 = T'_Q \oplus F_Q/F_Q$
(iii) $T_q N/T_q N_1 \cong Hom(k^2, k^3 \otimes V^*)/T'_Q \oplus F_Q$

Proof. (i) follows by a direct computation in linear algebra. Because the matrices of T'_q define a slice over N_1 by $Q + T'_Q$ and cover a neighbourhood of q in N_1, we obtain that $T'_Q \oplus F_Q/F_Q \subset T_Q N_1 \subset T'_Q \oplus F_Q/F_Q$. (iii) is a consequence of (ii). □

Remark: The above lemma implies also that N_1 is smooth of dimension 5 because $\dim T'_Q = 5$, $\dim F_Q = 12$. It is actually isomorphic to the tautological plane bundle over $\mathbb{P}_3^*$, in other words, to the flag manifold $\mathbb{F}lag(1,3,V^*)$.

8.4. **The morphism $M \xrightarrow{\rho} N$.**
Recalling the definition of the parameter space $X = X_0 \cup X_1$ of M, as a space of pairs (B, A) with

$$B = \left(\begin{array}{c|ccc} f & x_1 & x_2 & x_3 \\ g & y_1 & y_2 & y_3 \end{array} \right), \quad A = \left(\begin{array}{c|c} z & \lambda \\ \hline q_1 & z_1 \\ q_2 & z_2 \\ q_3 & z_3 \end{array} \right),$$

we know that the component X_1 is defined by the condition $\lambda = 0$. It follows that $M_1 \subset M$ and then also $M_0 \cap M_1 \subset M_0$ is a Cartier divisor in its reduced structure. Moreover, the submatrix $\left(\begin{smallmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{smallmatrix} \right)$ of B is stable, see last part of the proof of lemma 5.3. Thus

$$(B, A) \mapsto \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

is an equivariant morphism $X \to \mathrm{Hom}(k^2, k^3 \otimes V^*)^s$, which induces a morphism $M \xrightarrow{\rho} N$ of the geometric quotients. It maps M_1 and $M_0 \cap M_1$ onto N_1 as follows from the normal form of pairs $(B, A) \in X_1$, see proposition 3.5. Moreover, the restriction

$$M \smallsetminus M_1 \underset{\approx}{\xrightarrow{\rho}} N \smallsetminus N_1$$

is an isomorphism, using the normal form with $\lambda = 1$.

8.5. **Proposition.** *$M_0 \xrightarrow{\rho} N$ is isomorphic to the blowup $Bl_{N_1}N$ of N along N_1, and $M_0 \cap M_1$ is isomorphic to the projective normal bundle $\mathbb{P}(\mathcal{N}_{N_1/N})$.*

Proof. a) Because $M_0 \cap M_1$ is a Cartier divisor in M_0 which is mapped onto N_1, there is the unique natural morphism

$$\begin{tikzcd} M_0 \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & \mathrm{Bl}_{N_1}(N) \arrow[dl, "\sigma"] \\ & N & \end{tikzcd}$$

mapping $M_0 \cap M_1$ to the exceptional divisor $E \cong \mathbb{P}(\mathcal{N}_{N_1/N})$, where we use that N_1 is smooth. In our case $\varphi|_{M_0 \smallsetminus (M_0 \cap M_1)}$ is already an isomorphism. We are now going to prove that the induced morphism

$$M_0 \cap M_1 \xrightarrow{\tau} \mathbb{P}(\mathcal{N}_{N_1/N})$$

is an isomorphism. For that it is sufficient, that the induced map

$$(M_0 \cap M_1)_q \xrightarrow{\tau_q} \mathbb{P}(T_q N / T_q N_1)$$

is bijective for the fibers over a point $q \in N_1$. Then φ is bijective and an isomorphism because $\mathrm{Bl}_{N_1}(N)$ is smooth.

b) The map τ_q is defined as follows. For any point $[B_0, A_0] \in M_0 \cap M_1$ over q choose a tangent vector

$$\xi \in T_{[B_0,A_0]}M_0 \smallsetminus T_{[B_0,A_0]}M_1.$$

Its image in $T_qN \smallsetminus T_qN_1$ and then its image in $\mathbb{P}(T_qN/T_qN_1)$ is the image $\tau_q([B_0, A_0])$. More explicitly, if

$$q = \begin{bmatrix} l_1 & w & 0 \\ l_2 & 0 & w \end{bmatrix} \in N_1,$$

the points $[B_0, A_0]$ over q are of the form

$$B_0 = \begin{pmatrix} -q_1 & -l_1 & w & 0 \\ -q_2 & -l_2 & 0 & w \end{pmatrix}, \quad A_0 = \begin{pmatrix} w & 0 \\ 0 & w \\ q_1 & l_1 \\ q_2 & l_2 \end{pmatrix}$$

with $q_1, q_2 \in (l_1, l_2)$, see section 7. Then

$$q_1 = a_1 l_1 + b_1 l_2 \quad \text{and} \quad q_2 = a_2 l_1 + b_2 l_2$$

as in section 7, and for the tangent vector ξ we can choose the image of the pair

$$B_1 = \begin{pmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_1 + b_2 & 1 \\ b_1 a_2 - b_2 a_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

as in the proof of lemma 7.1. The image of this tangent vector in $\mathbb{P}(T_qN/T_qN_1)$ is simply the class of the matrix $\left(\begin{smallmatrix} 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \end{smallmatrix}\right)$. Hence, the map τ_q is given in notation of classes by

$$\left[\begin{pmatrix} -q_1 & -l_1 & w & 0 \\ -q_2 & -l_2 & 0 & w \end{pmatrix}, \begin{pmatrix} w & 0 \\ 0 & w \\ q_1 & l_1 \\ q_2 & l_2 \end{pmatrix} \right] \mapsto \overline{\langle \begin{pmatrix} 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \end{pmatrix} \rangle}.$$

It is now an easy exercise to verify that this map is well-defined and bijective. This completes the proof of proposition 8.5. □

8.6. **Corollary.** *M_0 is rational and smooth of dimension 12 and $M_0 \cap M_1$ is a smooth rational divisor in M_0.*

8.7. **Corollary.** *The components M_0 and M_1 intersect transversally in $M_0 \cap M_1$ and for any $p \in M_0 \cap M_1$*

$$T_pM_0 + T_pM_1 = T_pM$$

has dimension 14, *whereas* $\dim T_pM_0 \cap T_pM_1 = 11$.

9. The Chow groups of the moduli space

Let $k = \mathbb{C}$ and $V \cong k^4$. Using the descriptions of M_1 as the relative universal cubic $M_1 = \mathcal{Z} \to \mathbb{P}V^*$ and of M_0 as the blow up $\mathrm{Bl}_{N_1}(N)$ with exceptional divisor $M_0 \cap M_1 \cong \mathbb{P}(\mathcal{N}_{N_1/N})$, we determine now the Betti numbers of the Chow groups of the moduli space $M = M_{3m+1}(\mathbb{P}_3)$. Key ingredient is the following result on the Chow groups $A_*(N)$ of the space N of nets of quadrics, see [2]:

9.1. **Proposition** (Ellingsrud, Strømme). *The Betti numbers are*

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$b_i(N)$	1	1	3	4	7	8	10	8	7	4	3	1	1

In particular, the topological Euler characteristic is $e(N) = 58$.

9.2. **Theorem.** *The Chow groups of $M_{3m+1}(\mathbb{P}_3)$, of its components M_0, M_1 and of the intersection $M_0 \cap M_1$ are free and have the following topological Betti numbers b_i:*

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	$e(\cdot)$
$b_i(M_0 \cap M_1)$	1	3	6	9	11	12	12	11	9	6	3	1	–	–	84
$b_i(M_0)$	1	2	6	10	16	19	22	19	16	10	6	2	1	–	130
$b_i(M_1)$	1	3	6	9	11	12	12	12	12	11	9	6	3	1	108
$b_i(M_{3m+1}(\mathbb{P}_3))$	1	2	6	10	16	19	22	20	19	15	12	7	4	1	154

Furthermore, the Chow ring $A^*(M_1)$ is isomorphic to $\mathbb{Z}[s,t,u]/\mathfrak{a}$, where the ideal $\mathfrak{a}$ is given by

$$\mathfrak{a} = (s^4, t^3 - st^2 + s^2t - s^3, u^9 + (10\,s - 3\,t)u^8 + (55\,s^2 - 30\,st + 9\,t^2)u^7 + (220\,s^3 - 165\,s^2t + 90\,st^2)u^6 + \\ + (495\,s^2t^2 - 660\,s^3t)u^5 + 1980\,s^3t^2u^4).$$

Proof. Since $N_1 \cong \mathbb{F}lag(1,3,V^*)$ we obtain immediately

$$A^*(N_1) \cong A^*(\mathbb{P}V)[t]/(t^3 + c_1(\mathcal{Q})t^2 + c_2(\mathcal{Q})t + c_3(\mathcal{Q})) \cong \mathbb{Z}[s,t]/(s^4, s^3 + st^2 + s^2t + s^3),$$

where $\mathcal{Q}$ is the tautological quotient in the sequence $0 \to \mathcal{O}_{\mathbb{P}V}(-1) \to \mathcal{O}_{\mathbb{P}V} \otimes V^* \to \mathcal{Q} \to 0$. We get the Betti numbers for $M_0 \cap M_1$ using

$$A_i(M_0 \cap M_1) \cong A_i(\mathbb{P}(\mathcal{N}_{N_1/N})) \cong \bigoplus_{k=0}^{6} A_{i-6+k}(N_1)$$

and the presentation of the ring $A^*(N_1)$. But then it is straightforward to compute the numbers for M_0 from

$$\begin{tikzcd}
0 \arrow[r] & A_i(N_1) \arrow[r] \arrow[d, "\approx"] & A_i(\mathbb{P}(\mathcal{N}_{N_1/N})) \oplus A_i(N) \arrow[r] \arrow[d, "\approx"] & A_i(\mathrm{Bl}_{N_1}(N)) \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & A_i(N_1) \arrow[r] & A_i(M_0 \cap M_1) \oplus A_i(N) \arrow[r] & A_i(M_0) \arrow[r] & 0
\end{tikzcd}$$

since we know $b_i(N)$, $i = 1, \dots, 12$ from proposition 9.1.

Now recall the construction of the relative universal cubic $\mathcal{Z} \to \mathbb{P}V^*$ parametrizing triples (p, C, H) of points p on cubic curves C contained in planes $H \subset \mathbb{P}V^*$: Consider the exact sequence

$$0 \to \mathcal{H} \longrightarrow \mathcal{O}_{\mathbb{P}V^*} \otimes V \xrightarrow{eval} \mathcal{O}_{\mathbb{P}V^*}(1) \to 0.$$

The plane bundle $\mathbb{P}(\mathcal{H}) \xrightarrow{p} \mathbb{P}V^*$ comes with a surjective map $p^*\mathcal{H}^* \to \mathcal{O}_{\mathcal{H}}(1) \to 0$ which in turn induces

$$0 \to \mathcal{K} \longrightarrow p^* S^3\mathcal{H}^* \longrightarrow \mathcal{O}_{\mathcal{H}}(3) \to 0.$$

Then $\mathcal{Z} = \mathbb{P}(\mathcal{K}) \to \mathbb{P}(\mathcal{H}) \to \mathbb{P}V^*$ is the desired incidence variety.

Using this description of $\mathcal{Z}$, we compute the Chow ring of $M_1 \cong \mathcal{Z}$ as follows: The total Chern class of $\mathcal{H}$ is $c(\mathcal{H}) = 1 - s + s^2 - s^3$ and consequently

$$A^*(\mathbb{P}(\mathcal{H})) = \mathbb{Z}[s,t]/(s^4, t^3 - st^2 + s^2t - s^3).$$

One can easily check that $c(S^3\mathcal{H}^*) = 1 + 10\,s + 55\,s^2 + 220\,s^3$. Then $c(\mathcal{K}) = (1 + 10\,s + 55\,s^2 + 220\,s^3)(1+3t)^{-1}$ implies

$$A^*(M_1) \cong A^*(\mathbb{P}(\mathcal{H}))[u]/(u^9 + c_1(\mathcal{K})u^8 + \dots + c_9(\mathcal{K})) \cong \mathbb{Z}[s,t,u]/(s^4, t^3 - st^2 + s^2t - s^3, f)\,,$$

where $f = u^9 + (10\,s - 3\,t)u^8 + (55\,s^2 - 30\,st + 9\,t^2)u^7 + (220\,s^3 - 165\,s^2t + 90\,st^2)u^6 + (495\,s^2t^2 - 660\,s^3t)u^5 + 1980\,s^3t^2u^4$. This allows us also to determine the Betti numbers of M_1. Finally, since $k \cong \mathbb{C}$, we get the following diagram

$$\begin{tikzcd}
H_{2i+1}(M) \arrow[r] & H_{2i}(M_0 \cap M_1) \arrow[r] & H_{2i}(M_0) \oplus H_{2i}(M_1) \arrow[r] & H_{2i}(M) \arrow[r] & H_{2i-1}(M_0 \cap M_1) \\
0 \arrow[r] \arrow[u, "\approx"'] & A_i(M_0 \cap M_1) \arrow[r] \arrow[u, "cl", "\approx"'] & A_i(M_0) \oplus A_i(M_1) \arrow[r] \arrow[u, "cl", "\approx"'] & A_i(M) \arrow[r] \arrow[u, "cl", "\approx"'] & 0\,, \arrow[u, "\approx"']
\end{tikzcd}$$

where we use homology with locally compact supports. Note that in our case the cycle maps cl are isomorphisms. Using this Mayer–Vietoris sequence, we obtain immediately the Betti numbers of the moduli space $M = M_{3m+1}(\mathbb{P}_3)$. $\square$

References

[1] J.M. Drézet and G. Trautmann: *Moduli spaces of decomposable morphisms of sheaves and quotients by non–reductive groups.* Ann. Institute Fourier 53 (2003), 107-192.

[2] G. Ellingsrud and S.A. Strømme: *On the Chow ring of a geometric quotient.* Ann. Math., 130 (1989), 159-187.

[3] H.G. Freiermuth: *On the Moduli Space $M_P(\mathbb{P}_3)$ of Semi-stable Sheaves on $\mathbb{P}_3$ with Hilbert Polynomial $P(m) = 3m+1$.* Diplomarbeit, Universität Kaiserslautern, (December 2000).

[4] W. Fulton: *Intersection Theory.* Erg. der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 2, Springer (1984).

[5] J. Harris: *Curves in Projective Space.* Lecture Notes Montreal, SMS 35, Les Presses de l'Université de Montréal (1982).

[6] D. Huybrechts and M. Lehn: *The geometry of moduli spaces of sheaves.* Aspects Math., Vol. E 31, Vieweg (1997).

[7] J. Le Potier: *L'espace de modules de Simpson.* Preprint Séminaire de géométrie algébrique, Jussieu (1992).

[8] J. Le Potier: *Faisceaux semi-stables de dimension* 1 *sur le plan projectif.* Rev. Roumaine math. pures appl., 38 (1993), 7-8, 635-678.

[9] J. Fogarty and D. Mumford: *Geometric Invariant Theory.* Erg. der Mathematik und ihrer Grenzgebiete, Bd. 34, Springer (1982).

[10] C. Okonek, M. Schneider and H. Spindler: *Vector bundles on complex projective spaces.* Progr. Math. 3, Birkhäuser (1980).

[11] R. Piene and M. Schlessinger: *On the Hilbert scheme compactification of the space of twisted cubics.* Amer. J. Math., 107 (1985), 761-774.

[12] C.T. Simpson: *Moduli of representations of the fundamental group of a smooth projective variety I.* Publ. Math. I.H.E.S., 79 (1994), 47-129.

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027

E-mail address: freiermuth@math.columbia.edu

Fachbereich Mathematik, Universität Kaiserslautern, Erwin-Schrödinger Strasse, D-67663 Kaiserslautern

E-mail address: trm@mathematik.uni-kl.de